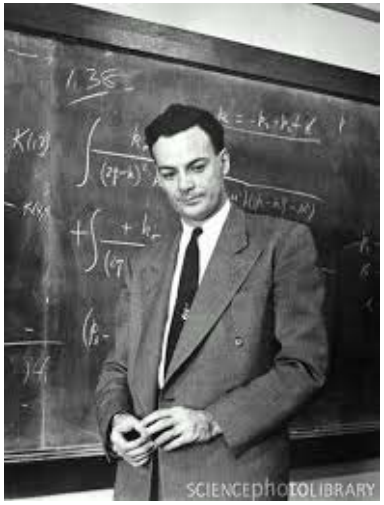
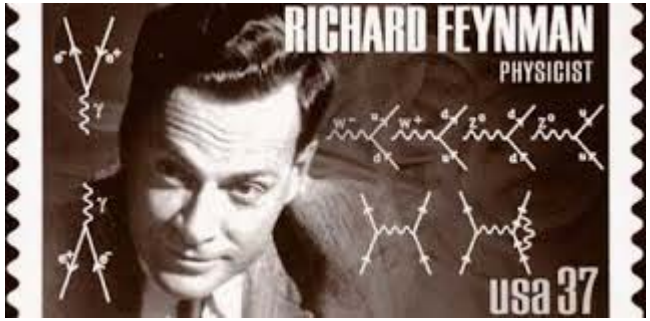
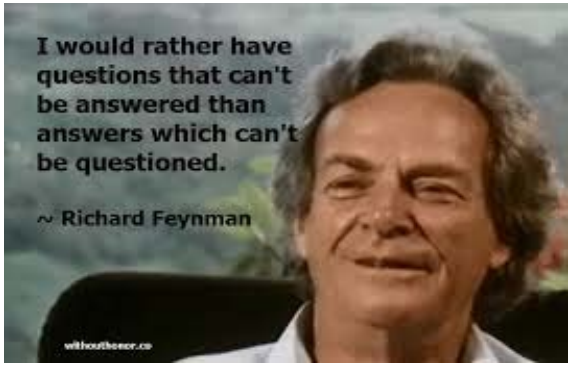
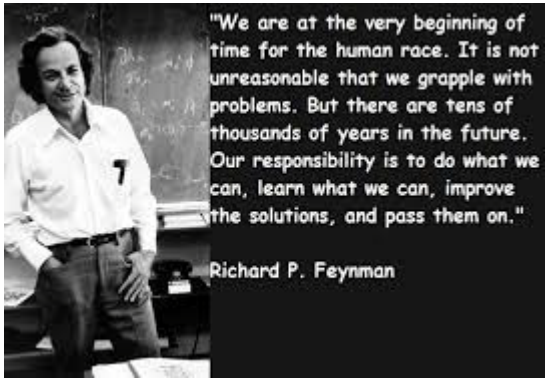
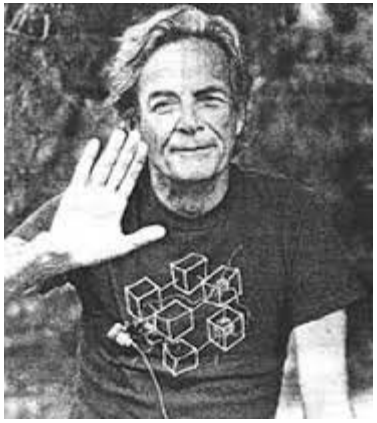


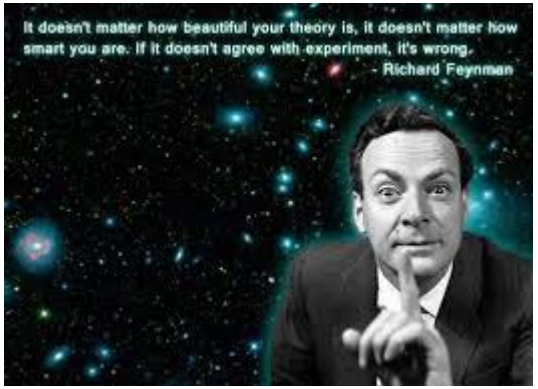
Richard Feynman
(1918-1988)











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Simulating Physics with Computers

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1. INTRODUCTION

On the program it says this is a keynote speech—and I don't know what a keynote speech is. I do not intend in any way to suggest what should be in this meeting as a keynote of the subjects or anything like that. I have my own things to say and to talk about and there's no implication that anybody needs to talk about the same thing or anything like it. So what I want to talk about is what Mike Dertouzos suggested that nobody would talk about. I want to talk about the problem of simulating physics with computers and I mean that in a specific way which I am going to explain. The reason for doing this is something that I learned about from Ed Fredkin, and my entire interest in the subject has been inspired by him. It has to do with learning something about the possibilities of computers, and also something about possibilities in physics. If we suppose that we know all the physical laws perfectly, of course we don't have to pay any attention to computers. It's interesting anyway to entertain oneself with the idea that we've got something to learn about physical laws; and if I take a relaxed view here (after all I'm here and not at home) I'll admit that we don't understand everything.

The first question is, What kind of computer are we going to use to simulate physics? Computer theory has been developed to a point where it realizes that it doesn't make any difference; when you get to a *universal computer*, it doesn't matter how it's manufactured, how it's actually made. Therefore my question is, Can physics be simulated by a universal computer? I would like to have the elements of this computer *locally interconnected*, and therefore sort of think about cellular automata as an example (but I don't want to force it). But I do want something involved with the

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locality of interaction. I would not like to think of a very enormous computer with arbitrary interconnections throughout the entire thing.

Now, what kind of physics are we going to imitate? First, I am going to describe the possibility of simulating physics in the classical approximation, a thing which is usually described by local differential equations. But the physical world is quantum mechanical, and therefore the proper problem is the simulation of quantum physics—which is what I really want to talk about, but I'll come to that later. So what kind of simulation do I mean? There is, of course, a kind of approximate simulation in which you design numerical algorithms for differential equations, and then use the computer to compute these algorithms and get an approximate view of what physics ought to do. That's an interesting subject, but is not what I want to talk about. I want to talk about the possibility that there is to be an *exact* simulation, that the computer will do *exactly* the same as nature. If this is to be proved and the type of computer is as I've already explained, then it's going to be necessary that *everything* that happens in a finite volume of space and time would have to be exactly analyzable with a finite number of logical operations. The present theory of physics is not that way, apparently. It allows space to go down into infinitesimal distances, wavelengths to get infinitely great, terms to be summed in infinite order, and so forth; and therefore, if this proposition is right, physical law is wrong.

So good, we already have a suggestion of how we might modify physical law, and that is the kind of reason why I like to study this sort of problem. To take an example, we might change the idea that space is continuous to the idea that space perhaps is a simple lattice and everything is discrete (so that we can put it into a finite number of digits) and that time jumps discontinuously. Now let's see what kind of a physical world it would be or what kind of problem of computation we would have. For example, the first difficulty that would come out is that the speed of light would depend slightly on the direction, and there might be other anisotropies in the physics that we could detect experimentally. They might be very small anisotropies. Physical knowledge is of course always incomplete, and you can always say we'll try to design something which beats experiment at the present time, but which predicts anistropies on some scale to be found later. That's fine. That would be good physics if you could predict something consistent with all the known facts and suggest some new fact that we didn't explain, but I have no specific examples. So I'm not objecting to the fact that it's anistropic in principle, it's a question of how anistropic. If you tell me it's so-and-so anistropic, I'll tell you about the experiment with the lithium atom which shows that the anistropy is less than that much, and that this here theory of yours is impossible.

Another thing that had been suggested early was that natural laws are reversible, but that computer rules are not. But this turned out to be false; the computer rules can be reversible, and it has been a very, very useful thing to notice and to discover that. (Editors' note: see papers by Bennett, Fredkin, and Toffoli, these Proceedings). This is a place where the relationship of physics and computation has turned itself the other way and told us something about the possibilities of computation. So this is an interesting subject because it tells us something about computer rules, and *might* tell us something about physics.

The rule of simulation that I would like to have is that the number of computer elements required to simulate a large physical system is only to be proportional to the space-time volume of the physical system. I don't want to have an explosion. That is, if you say I want to explain this much physics, I can do it exactly and I need a certain-sized computer. If doubling the volume of space and time means I'll need an *exponentially* larger computer, I consider that against the rules (I make up the rules, I'm allowed to do that). Let's start with a few interesting questions.

2. SIMULATING TIME

First I'd like to talk about simulating time. We're going to assume it's discrete. You know that we don't have infinite accuracy in physical measurements so time might be discrete on a scale of less than 10^{-27} sec. (You'd have to have it at least like to this to avoid clashes with experiment—but make it 10^{-41} sec. if you like, and then you've got us!)

One way in which we simulate time—in cellular automata, for example —is to say that "the computer goes from state to state." But really, that's using intuition that involves the idea of time—you're going from state to state. And therefore the time (by the way, like the space in the case of cellular automata) is not simulated at all, it's imitated in the computer.

An interesting question comes up: "Is there a way of simulating it, rather than imitating it?" Well, there's a way of looking at the world that is called the space-time view, imagining that the points of space and time are all laid out, so to speak, ahead of time. And then we could say that a "computer" rule (now computer would be in quotes, because it's not the standard kind of computer which cperates in time) is: We have a state s_i at each point i in space-time. (See Figure 1.) The state s_i at the space time point i is a given function $F_i(s_j, s_k, \ldots)$ of the state at the points j, k in some neighborhood of i:

$$s_i = F_i(s_j,, s_k, \ldots)$$

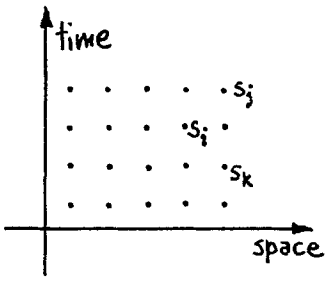

Fig. 1.

You'll notice immediately that if this particular function is such that the value of the function at i only involves the few points behind in time, earlier than this time i, all I've done is to redescribe the cellular automaton, because it means that you calculate a given point from points at earlier times, and I can compute the next one and so on, and I can go through this in that particular order. But just let's us think of a more general kind of computer, because we might have a more general function. So let's think about whether we could have a wider case of generality of interconnections of points in space-time. If F depends on *all* the points both in the future and the past, what then? That could be the way physics works. I'll mention how our theories go at the moment. It has turned out in many physical theories that the mathematical equations are quite a bit simplified by imagining such a thing—by imagining positrons as electrons going backwards in time, and other things that connect objects forward and backward. The important question would be, if this computer were laid out, is there in fact an organized algorithm by which a solution could be laid out, that is, computed? Suppose you know this function F_i and it is a function of the variables in the future as well. How would you lay out numbers so that they automatically satisfy the above equation? It may not be possible. In the case of the cellular automaton it is, because from a given row you get the next row and then the next row, and there's an organized way of doing it. It's an interesting question whether there are circumstances where you get functions for which you can't think, at least right away, of an organized way of laying it out. Maybe sort of shake it down from some approximation, or something, but it's an interesting different type of computation.

Question: "Doesn't this reduce to the ordinary boundary value, as opposed to initial-value type of calculation?"

Answer: "Yes, but remember this is the computer itself that I'm describing."

It appears actually that classical physics is causal. You can, in terms of the information in the past, if you include both momentum and position, or

the position at two different times in the past (either way, you need two pieces of information at each point) calculate the future in principle. So classical physics is *local*, *causal*, and *reversible*, and therefore apparently quite adaptable (except for the discreteness and so on, which I already mentioned) to computer simulation. We have no difficulty, in principle, apparently, with that.

3. SIMULATING PROBABILITY

Turning to quantum mechanics, we know immediately that here we get only the ability, apparently, to predict probabilities. Might I say immediately, so that you know where I really intend to go, that we always have had (secret, secret, close the doors!) we always have had a great deal of difficulty in understanding the world view that quantum mechanics represents. At least I do, because I'm an old enough man that I haven't got to the point that this stuff is obvious to me. Okay, I still get nervous with it. And therefore, some of the younger students ... you know how it always is, every new idea, it takes a generation or two until it becomes obvious that there's no real problem. It has not yet become obvious to me that there's no real problem. I cannot define the real problem, therefore I suspect there's no real problem, but I'm note sure there's no real problem. So that's why I like to investigate things. Can I learn anything from asking this question about computers—about this may or may not be mystery as to what the world view of quantum mechanics is? So I know that quantum mechanics seem to involve probability—and I therefore want to talk about simulating probability.

Well, one way that we could have a computer that simulates a probabilistic theory, something that has a probability in it, would be to calculate the probability and then interpret this number to represent nature. For example, let's suppose that a particle has a probability $P(x, t)$ to be at x at a time t. A typical example of such a probability might satisfy a differential equation, as, for example, if the particle is diffusing:

$$\frac{\partial P(x,t)}{\partial t} = -\nabla^2 P(x,t)$$

Now we could discretize t and x and perhaps even the probability itself and solve this differential equation like we solve any old field equation, and make an algorithm for it, making it exact by discretization. First there'd be a problem about discretizing probability. If you are only going to take k digits it would mean that when the probability is less that 2^{-k} of something happening, you say it doesn't happen at all. In practice we do that. If the

probability of something is 10^{-700}, we say it isn't going to happen, and we're not caught out very often. So we could allow ourselves to do that. But the real difficulty is this: If we had many particles, we have R particles, for example, in a system, then we would have to describe the probability of a circumstance by giving the probability to find these particles at points $x_1, x_2, \ldots, x_R$ at the time t. That would be a description of the probability of the system. And therefore, you'd need a k-digit number for every configuration of the system, for every arrangement of the R values of x. And therefore if there are N points in space, we'd need N^R configurations. Actually, from our point of view that at each point in space there is information like electric fields and so on, R will be of the same order as N if the number of information bits is the same as the number of points in space, and therefore you'd have to have something like N^N configurations to be described to get the probability out, and that's too big for our computer to hold if the size of the computer is of order N.

We emphasize, if a description of an isolated part of nature with N variables requires a general function of N variables and if a computer stimulates this by actually computing or storing this function then doubling the size of nature ($N \rightarrow 2N$) would require an exponentially explosive growth in the size of the simulating computer. It is therefore impossible, according to the rules stated, to simulate by calculating the probability.

Is there any other way? What kind of simulation can we have? We can't expect to compute the probability of configurations for a probabilistic theory. But the other way to simulate a probabilistic nature, which I'll call $\mathcal{N}$ for the moment, might still be to simulate the probabilistic nature by a computer $\mathcal{C}$ which itself is probabilistic, in which you always randomize the last two digit's of every number, or you do something terrible to it. So it becomes what I'll call a probabilistic computer, in which the output is not a unique function of the input. And then you try to work it out so that it simulates nature in this sense: that $\mathcal{C}$ goes from some state—initial state if you like—to some final state with the *same* probability that $\mathcal{N}$ goes from the corresponding initial state to the corresponding final state. Of course when you set up the machine and let nature do it, the imitator will not do the same thing, it only does it with the same probability. Is that no good? No it's O.K. How do you know what the probability is? You see, nature's unpredictable; how do you expect to predict it with a computer? You can't, —it's unpredictable if it's probabilistic. But what you really do in a probabilistic system is repeat the experiment in nature a large number of times. If you repeat the same experiment in the computer a large number of times (and that doesn't take any more time than it does to do the same thing in nature of course), it will give the frequency of a given final state proportional to the number of times, with approximately the same rate (plus

or minus the square root of n and all that) as it happens in nature. In other words, we could imagine and be perfectly happy, I think, with a probabilistic simulator of a probabilistic nature, in which the machine doesn't exactly do what nature does, but if you repeated a particular type of experiment a sufficient number of times to determine nature's probability, then you did the corresponding experiment on the computer, you'd get the corresponding probability with the corresponding accuracy (with the same kind of accuracy of statistics).

So let us now think about the characteristics of a local probabilistic computer, because I'll see if I can imitate nature with that (by "nature" I'm now going to mean quantum mechanics). One of the characteristics is that you can determine how it behaves in a local region by simply disregarding what it's doing in all other regions. For example, suppose there are variables in the system that describe the whole world (x_A, x_B)—the variables x_A you're interested in, they're "around here"; x_B are the whole result of the world. If you want to know the probability that something around here is happening, you would have to get that by integrating the total probability of all kinds of possibilities over x_B. If we had *computed* this probability, we would still have to do the integration

$$P_A(x_A) = \int P(x_A, x_B) dx_B$$

which is a hard job! But if we have *imitated* the probability, it's very simple to do it: you don't have to do anything to do the integration, you simply disregard what the values of x_B are, you just look at the region x_A. And therefore it does have the characteristic of nature: if it's local, you can find out what's happening in a region not by integrating or doing an extra operation, but merely by disregarding what happens elsewhere, which is no operation, nothing at all.

The other aspect that I want to emphasize is that the equations will have a form, no doubt, something like the following. Let each point $i = 1, 2, \ldots, N$ in space be in a state s_i chosen from a small state set (the size of this set should be reasonable, say, up to 2^5). And let the probability to find some configuration $\{s_i\}$ (a set of values of the state s_i at each point i) be some number $P(\{s_i\})$. It satisfies an equation such that at each jump in time

$$P_{t+1}(\{s\}) = \sum_{\{s'\}} \left[\prod_i m(s_i | s'_j, s'_k \ldots) \right] P_t(\{s'\})$$

where $m(s_i | s'_j, s'_k \ldots)$ is the probability that we move to state s_i at point i

when the neighbors have values $s'_j, s'_k \ldots$, where j, k etc. are points in the neighborhood of i. As j moves far from i, m becomes ever less sensitive to s'_j. At each change the state at a particular point i will move from what it was to a state s with a probability m that depends only upon the states of the neighborhood (which may be so defined as to include the point i itself). This gives the probability of making a transition. It's the same as in a cellular automaton; only, instead of its being definite, it's a probability. Tell me the environment, and I'll tell you the probability after a next moment of time that this point is at state s. And that's the way it's going to work, okay? So you get a mathematical equation of this kind of form.

Now I explicitly go to the question of how we can simulate with a computer—a universal automaton or something—the quantum-mechanical effects. (The usual formulation is that quantum mechanics has some sort of a differential equation for a function ψ.) If you have a single particle, ψ is a function of x and t, and this differential equation could be simulated just like my probabilistic equation was before. That would be all right and one has seen people make little computers which simulate the Schröedinger equation for a single particle. But the full description of quantum mechanics for a large system with R particles is given by a function $\psi(x_1, x_2, \ldots, x_R, t)$ which we call the amplitude to find the particles $x_1, \ldots, x_R$, and therefore, because it has too many variables, it *cannot be simulated* with a normal computer with a number of elements proportional to R or proportional to N. We had the same troubles with the probability in classical physics. And therefore, the problem is, how can we simulate the quantum mechanics? There are two ways that we can go about it. We can give up on our rule about what the computer was, we can say: Let the computer itself be built of quantum mechanical elements which obey quantum mechanical laws. Or we can turn the other way and say: Let the computer still be the same kind that we thought of before—a logical, universal automaton; can we imitate this situation? And I'm going to separate my talk here, for it branches into two parts.

4. QUANTUM COMPUTERS—UNIVERSAL QUANTUM SIMULATORS

The first branch, one you might call a side-remark, is, Can you do it with a new kind of computer—a quantum computer? (I'll come back to the other branch in a moment.) Now it turns out, as far as I can tell, that you can simulate this with a quantum system, with quantum computer elements. It's not a Turing machine, but a machine of a different kind. If we disregard the continuity of space and make it discrete, and so on, as an approximation (the same way as we allowed ourselves in the classical case), it does seem to

be true that all the various field theories have the same *kind* of behavior, and can be simulated in every way, apparently, with little latticeworks of spins and other things. It's been noted time and time again that the phenomena of field theory (if the world is made in a discrete lattice) are well imitated by many phenomena in solid state theory (which is simply the analysis of a latticework of crystal atoms, and in the case of the kind of solid state I mean each atom is just a point which has numbers associated with it, with quantum-mechanical rules). For example, the spin waves in a spin lattice imitating Bose-particles in the field theory. I therefore believe it's true that with a suitable class of quantum machines you could imitate any quantum system, including the physical world. But I don't know whether the general theory of this intersimulation of quantum systems has ever been worked out, and so I present that as another interesting problem: to work out the classes of different kinds of quantum mechanical systems which are really intersimulatable—which are equivalent—as has been done in the case of classical computers. It has been found that there is a kind of universal computer that can do anything, and it doesn't make much difference specifically how it's designed. The same way we should try to find out what kinds of quantum mechanical systems are mutually intersimulatable, and try to find a specific class, or a character of that class which will simulate everything. What, in other words, is the universal quantum simulator? (assuming this discretization of space and time). If you had discrete quantum systems, what other discrete quantum systems are exact imitators of it, and is there a class against which everything can be matched? I believe it's rather simple to answer that question and to find the class, but I just haven't done it.

Suppose that we try the following guess: that every finite quantum mechanical system can be described *exactly*, imitated exactly, by supposing that we have another system such that at each point in space-time this system has only two possible base states. Either that point is occupied, or unoccupied—those are the two states. The mathematics of the quantum mechanical operators associated with that point would be very simple.

$$a = \text{ANNIHILATE} = \begin{array}{c|cc} & \text{OCC} & \text{UN} \\ \hline \text{OCC} & 0 & 0 \\ \text{UN} & 1 & 0 \end{array} = \tfrac{1}{2}(\sigma_x - i\sigma_y)$$

$$a^* = \text{CREATE} = \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} = \tfrac{1}{2}(\sigma_x + i\sigma_y)$$

$$n = \text{NUMBER} = \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} = a^*a = \tfrac{1}{2}(1 + \sigma_z)$$

$$\mathbb{1} = \text{IDENTITY} = \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}$$

There would be an operator a which *annihilates* if the point is occupied —it changes it to unoccupied. There is a conjugate operator a^* which does the opposite: if it's unoccupied, it occupies it. There's another operator n called the *number* to ask, Is something there? The little matrices tell you what they do. If it's there, n gets a one and leaves it alone, if it's not there, nothing happens. That's mathematically equivalent to the product of the other two, as a matter of fact. And then there's the identity, $\mathbb{1}$, which we always have to put in there to complete our mathematics—it doesn't do a damn thing!

By the way, on the right-hand side of the above formulas the same operators are written in terms of matrices that most physicists find more convenient, because they are Hermitian, and that seems to make it easier for them. They have invented another set of matrices, the Pauli σ matrices:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And these are called *spin*—spin one-half—so sometimes people say you're talking about a spin-one-half lattice.

The question is, if we wrote a Hamiltonian which involved only these operators, locally coupled to corresponding operators on the other space-time points, could we imitate every quantum mechanical system which is discrete and has a finite number of degrees of freedom? I know, almost certainly, that we could do that for any quantum mechanical system which involves Bose particles. I'm not sure whether Fermi particles could be described by such a system. So I leave that open. Well, that's an example of what I meant by a general quantum mechanical simulator. I'm not sure that it's sufficient, because I'm not sure that it takes care of Fermi particles.

5. CAN QUANTUM SYSTEMS BE PROBABILISTICALLY SIMULATED BY A CLASSICAL COMPUTER?

Now the next question that I would like to bring up is, of course, the interesting one, i.e., Can a quantum system be probabilistically simulated by a classical (probabilistic, I'd assume) universal computer? In other words, a computer which will give the same probabilities as the quantum system does. If you take the computer to be the classical kind I've described so far, (not the quantum kind described in the last section) and there're no changes in any laws, and there's no hocus-pocus, the answer is certainly, No! This is called the hidden-variable problem: it is impossible to represent the results of quantum mechanics with a classical universal device. To learn a little bit about it, I say let us try to put the quantum equations in a form as close as

possible to classical equations so that we can see what the difficulty is and what happens. Well, first of all we can't simulate ψ in the normal way. As I've explained already, there're too many variables. Our only hope is that we're going to simulate probabilities, that we're going to have our computer do things with the same probability as we observe in nature, as calculated by the quantum mechanical system. Can you make a cellular automaton, or something, imitate with the same probability what nature does, where I'm going to suppose that quantum mechanics is correct, or at least after I discretize space and time it's correct, and see if I can do it. I must point out that you must directly generate the probabilities, the results, with the correct quantum probability. Directly, because we have no way to store all the numbers, we have to just imitate the phenomenon directly.

It turns out then that another thing, rather than the wave function, a thing called the *density matrix*, is much more useful for this. It's not so useful as far as the mathematical equations are concerned, since it's more complicated than the equations for ψ, but I'm not going to worry about mathematical complications, or which is the easiest way to calculate, because with computers we don't have to be so careful to do it the very easiest way. And so with a slight increase in the complexity of the equations (and not very much increase) I turn to the density matrix, which for a single particle of coordinate x in a pure state of wave function $\psi(x)$ is

$$\rho(x, x') = \psi^*(x)\psi(x')$$

This has a special property that is a function of two coordinates x, x'. The presence of two quantities x and x' associated with each coordinate is analogous to the fact that in classical mechanics you have to have two variables to describe the state, x and $\dot{x}$. States are described by a second-order device, with two informations ("position" and "velocity"). So we have to have two pieces of information associated with a particle, analogous to the classical situation, in order to describe configurations. (I've written the density matrix for one particle, but of course there's the analogous thing for R particles, a function of $2R$ variables).

This quantity has many of the mathematical properties of a probability. For example if a state $\psi(x)$ is not certain but is ψ_α with the probability p_α then the density matrix is the appropriate weighted sum of the matrix for each state α:

$$\rho(x, x') = \sum_\alpha p_\alpha \psi_\alpha^*(x)\psi a(x').$$

A quantity which has properties even more similar to classical probabilities is the Wigner function, a simple reexpression of the density matrix; for a

single particle

$$W(x,p)=\int \rho\left(x+\frac{y}{2},x-\frac{y}{2}\right)e^{ipy}dy$$

We shall be emphasizing their similarity and shall call it "probability" in quotes instead of Wigner function. Watch these quotes carefully, when they are absent we mean the real probability. If "probability" had all the mathematical properties of a probability we could remove the quotes and simulate it. $W(x, p)$ is the "probability" that the particle has position x and momentum p (per dx and dp). What properties does it have that are analogous to an ordinary probability?

It has the property that if there are many variables and you want to know the "probabilities" associated with a finite region, you simply disregard the other variables (by integration). Furthermore the probability of finding a particle at x is $\int W(x,p)dp$. If you can interpret W as a probability of finding x and p, this would be an expected equation. Likewise the probability of p would be expected to be $\int W(x,p)dx$. These two equations are correct, and therefore you would hope that maybe $W(x, p)$ is the probability of finding x and p. And the question then is can we make a device which simulates this W? Because then it would work fine.

Since the quantum systems I noted were best represented by spin one-half (occupied versus unoccupied or spin one-half is the same thing), I tried to do the same thing for spin one-half objects, and it's rather easy to do. Although before one object only had two states, occupied and unoccupied, the full description—in order to develop things as a function of time—requires twice as many variables, which mean two slots at each point which are occupied or unoccupied (denoted by + and − in what follows), analogous to the x and $\dot{x}$, or the x and p. So you can find four numbers, four "probabilities" $\{f_{++}, f_{+-}, f_{-+}, f_{--}\}$ which act just like, and I have to explain why they're not exactly like, but they act just like, probabilities to find things in the state in which both symbols are up, one's up and one's down, and so on. For example, the sum $f_{++}+f_{+-}+f_{-+}+f_{--}$ of the four "probabilities" is 1. You'll remember that one object now is going to have two indices, two plus/minus indices, or two ones and zeros at each point, although the quantum system had only one. For example, if you would like to know whether the first index is positive, the probability of that would be

$$\text{Prob(first index is +)}=f_{++}+f_{+-} \qquad [\text{spin } z \text{ up}]$$

i.e., you don't care about the second index. The probability that the first index is negative is

$$\text{Prob(first index is −)}=f_{-+}+f_{--}, \qquad [\text{spin } z \text{ down}]$$

These two formulas are exactly correct in quantum mechanics. You see I'm hedging on whether or not "probability" f can really be a probability without quotes. But when I write probability without quotes on the left-hand side I'm not hedging; that really is the quantum mechanical probability. It's interpreted perfectly fine here. Likewise the probability that the second index is positive can be obtained by finding

$$\text{Prob(second index is } +) = f_{++} + f_{-+} \qquad [\text{spin } x \text{ up}]$$

and likewise

$$\text{Prob(second index is } -) = f_{+-} + f_{--} \qquad [\text{spin } x \text{ down}]$$

You could also ask other questions about the system. You might like to know, What is the probability that both indices are positive? You'll get in trouble. But you could ask other questions that you won't get in trouble with, and that get correct physical answers. You can ask, for example, what is the probability that the two indices are the same? That would be

$$\text{Prob(match)} = f_{++} + f_{--} \qquad [\text{spin } y \text{ up}]$$

Or the probability that there's no match between the indices, that they're different,

$$\text{Prob(no match)} = f_{+-} + f_{-+} \qquad [\text{spin } y \text{ down}]$$

All perfectly all right. All these probabilities are correct and make sense, and have a precise meaning in the spin model, shown in the square brackets above. There are other "probability" combinations, other linear combinations of these f's which also make physically sensible probabilities, but I won't go into those now. There are other linear combinations that you can ask questions about, but you don't seem to be able to ask questions about an individual f.

6. NEGATIVE PROBABILITIES

Now, for many interacting spins on a lattice we can give a "probability" (the quotes remind us that there is still a question about whether it's a probability) for correlated possibilities:

$$F(s_1, s_2, \ldots, s_N) \qquad (s_i \in \{++, +-, -+, --\})$$

Next, if I look for the quantum mechanical equation which tells me what the changes of F are with time, they are exactly of the form that I wrote above for the classical theory:

$$F_{t+1}(\{s\}) = \sum_{\{s'\}} \left[\prod_i M(s_i | s'_j, s'_k \ldots) \right] F_t(\{s'\})$$

but now we have F instead of P. The $M(s_i|s'_j, s'_k \ldots)$ would appear to be interpreted as the "probability" per unit time, or per time jump, that the state at i turns into s_i when the neighbors are in configuration s'. If you can invent a probability M like that, you write the equations for it according to normal logic, those are the correct equations, the real, correct, quantum mechanical equations for this F, and therefore you'd say, Okay, so I can imitate it with a probabilistic computer!

There's only one thing wrong. These equations unfortunately cannot be so interpreted on the basis of the so-called "probability", or this probabilistic computer can't simulate them, because the F is not necessarily positive. Sometimes it's negative! The M, the "probability" (so-called) of moving from one condition to another is itself not positive; if I had gone all the way back to the f for a single object, it again is not necessarily positive.

An example of possibilities here are

$$f_{++} = 0.6 \qquad f_{+-} = -0.1 \qquad f_{-+} = 0.3 \qquad f_{--} = 0.2$$

The sum $f_{++} + f_{+-}$ is 0.5, that's 50% chance of finding the first index positive. The probability of finding the first index negative is the sum $f_{-+} + f_{-+}$ which is also 50%. The probability of finding the second index positive is the sum $f_{++} + f_{-+}$ which is nine tenths, the probability of finding it negative is $f_{+-} + f_{--}$ which is one-tenth, perfectly alright, it's either plus or minus. The probability that they match is eight-tenths, the probability that they mismatch is plus two-tenths; every physical probability comes out positive. But the original f's are not positive, and therein lies the great difficulty. The only difference between a probabilistic classical world and the equations of the quantum world is that somehow or other it appears as if the probabilities would have to go negative, and that we do not know, as far as I know, how to simulate. Okay, that's the fundamental problem. I don't know the answer to it, but I wanted to explain that if I try my best to make the equations look as near as possible to what would be imitable by a classical probabilistic computer, I get into trouble.

7. POLARIZATION OF PHOTONS—TWO-STATES SYSTEMS

I would like to show you why such minus signs cannot be avoided, or at least that you have some sort of difficulty. You probably have all heard this example of the Einstein-Podolsky-Rosen paradox, but I will explain this little example of a physical experiment which can be done, and which has been done, which does give the answers quantum theory predicts, and the answers are really right, there's no mistake, if you do the experiment, it actually comes out. And I'm going to use the example of polarizations of photons, which is an example of a two-state system. When a photon comes, you can say it's either x polarized or y polarized. You can find that out by putting in a piece of calcite, and the photon goes through the calcite either out in one direction, or out in another—actually slightly separated, and then you put in some mirrors, that's not important. You get two beams, two places out, where the photon can go. (See Figure 2.)

If you put a polarized photon in, then it will go to one beam called the ordinary ray, or another, the extraordinary one. If you put detectors there you find that each photon that you put in, it either comes out in one or the other 100% of the time, and not half and half. You either find a photon in one or the other. The probability of finding it in the ordinary ray plus the probability of finding it in the extraordinary ray is always 1—you have to have that rule. That works. And further, it's never found at both detectors. (If you might have put two photons in, you could get that, but you cut the intensity down—it's a technical thing, you don't find them in both detectors.)

Now the next experiment: Separation into 4 polarized beams (see Figure 3). You put two calcites in a row so that their axes have a relative angle ϕ, I happen to have drawn the second calcite in two positions, but it doesn't make a difference if you use the same piece or not, as you care. Take the ordinary ray from one and put it through another piece of calcite and look at its ordinary ray, which I'll call the ordinary–ordinary ($O-O$) ray, or look at its extraordinary ray, I have the ordinary–extraordinary ($O-E$) ray. And then the extraordinary ray from the first one comes out as the $E-O$ ray, and then there's an $E-E$ ray, alright. Now you can ask what happens.

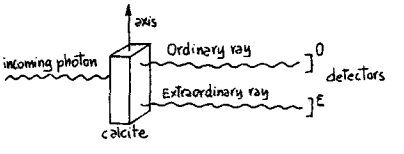

Fig. 2.

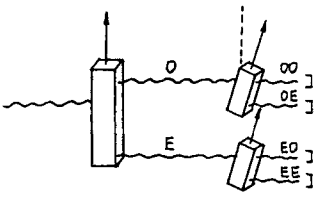

Fig. 3.

You'll find the following. *When a photon comes in, you always find that only one of the four counters goes off.*

If the photon is O from the first calcite, then the second calcite gives $O-O$ with probability $\cos^2\phi$ or $O-E$ with the complementary probability $1-\cos^1\phi=\sin^2\phi$. Likewise an E photon gives a $E-O$ with the probability $\sin^2\phi$ or an $E-E$ with the probability $\cos^2\phi$.

8. TWO-PHOTON CORRELATION EXPERIMENT

Let us turn now to the two photon correlation experiment (see Figure 4).

What can happen is that an atom emits two photons in opposite direction (e.g., the $3s\rightarrow 2p\rightarrow 1s$ transition in the H atom). They are observed simultaneously (say, by you and by me) through two calcites set at ϕ_1 and ϕ_2 to the vertical. Quantum theory and experiment agree that the probability P_{OO} that both of us detect an ordinary photon is

$$P_{OO}=\tfrac{1}{2}\cos^2(\phi_2-\phi_1)$$

The probability P_{EE} that we both observe an extraordinary ray is the same

$$P_{EE}=\tfrac{1}{2}\cos^2(\phi_2-\phi_1)$$

The probability P_{OE} that I find O and you find E is

$$P_{OE}=\tfrac{1}{2}\sin^2(\phi_2-\phi_1)$$

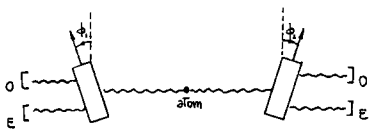

Fig. 4.

and finally the probability P_{EO} that I measure E and you measure O is

$$P_{EO} = \tfrac{1}{2}\sin^2(\phi_2 - \phi_1)$$

Notice that you can always predict, from your own measurement, what I shall get, O or E. For any axis ϕ_1 that I chose, just set your axis ϕ_2 to ϕ_1, then

$$P_{OE} = P_{EO} = 0$$

and I must get whatever you get.

Let us see now how it would have to be for a *local* probabilistic computer. Photon 1 must be in some condition α with the probability $f_\alpha(\phi_1)$, that determines it to go through as an ordinary ray [the probability it would pass as E is $1 - f_\alpha(\phi_1)$]. Likewise photon 2 will be in a condition β with probability $g_\beta(\phi_2)$. If $p_{\alpha\beta}$ is the conjoint probability to find the condition pair α, β, the probability P_{OO} that both of us observe O rays is

$$P_{OO}(\phi_1, \phi_2) = \sum_{\alpha\beta} p_{\alpha\beta} f_\alpha(\phi_1) g_\beta(\phi_2) \qquad \sum_{\alpha\beta} p_{\alpha\beta} = 1$$

likewise

$$P_{OE}(\phi_1, \phi_2) = \sum_{\alpha\beta} p_{\alpha\beta}(1 - f_\alpha(\phi_1)) g_\beta(\phi_2) \quad \text{etc.}$$

The conditions α determine how the photons go. There's some kind of correlation of the conditions. Such a formula cannot reproduce the quantum results above for any $p_{\alpha\beta}$, $f_\alpha(\phi_1)$, $g_\beta(\phi_2)$ if they are real probabilities—that is all positive, although it is easy if they are "probabilities"—negative for some conditions or angles. We now analyze why that is so.

I don't know what kinds of conditions they are, but for any condition the probability $f_\alpha(\phi)$ of its being extraordinary or ordinary in any direction must be either one or zero. Otherwise you couldn't predict it on the other side. You would be unable to predict with certainty what I was going to get, unless, every time the photon comes here, which way it's going to go is absolutely determined. Therefore, whatever condition the photon is in, there is some hidden inside variable that's going to determine whether it's going to be ordinary or extraordinary. This determination is done deterministically, not probabilistically; otherwise we can't explain the fact that you could predict what I was going to get *exactly*. So let us suppose that something like this happens. Suppose we discuss results just for angles which are multiples of 30°.

On each diagram (Figure 5) are the angles 0°, 30°, 60°, 90°, 120°, and 150°. A particle comes out to me, and it's in some sort of state, so what it's going to give for 0°, for 30°, etc. are all predicted—determined—by the state. Let us say that in a particular state that is set up the prediction for 0° is that it'll be extraordinary (black dot), for 30° it's also extraordinary, for 60° it's ordinary (white dot), and so on (Figure 5a). By the way, the outcomes are complements of each other at right angles, because, remember, it's always either extraordinary or ordinary; so if you turn 90°, what used to be an ordinary ray becomes the extraordinary ray. Therefore, whatever condition it's in, it has some predictive pattern in which you either have a prediction of ordinary or of extraordinary—three and three—because at right angles they're not the same color. Likewise the particle that comes to you when they're separated must have the same pattern because you can determine what I'm going to get by measuring yours. Whatever circumstances come out, the patterns must be the same. So, if I want to know, Am I going to get white at 60°? You just measure at 60°, and you'll find white, and therefore you'll predict white, or ordinary, for me. Now each time we do the experiment the pattern may not be the same. Every time we make a pair of photons, repeating this experiment again and again, it doesn't have to be the same as Figure 5a. Let's assume that the next time the experiment my photon will be *O* or *E* for each angle as in Figure 5c. Then your pattern looks like Figure 5d. But whatever it is, your pattern has to be my pattern exactly—otherwise you couldn't predict what I was going to get exactly by measuring the corresponding angle. And so on. Each time we do the experiment, we get different patterns; and it's easy: there are just six dots and three of them are white, and you chase them around different way—everything can happen. If we measure at the same angle, we always find that with this kind of arrangement we would get the same result.

Now suppose we measure at $\phi_2 - \phi_1 = 30°$, and ask, With what probability do we get the same result? Let's first try this example here (Figure 5a, 5b). With what probability would we get the same result, that they're

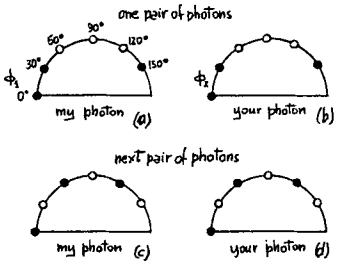

Fig. 5.

both white, or they're both black? The thing comes out like this: suppose I say, After they come out, I'm going to choose a direction at random, I tell you to measure 30° to the right of that direction. Then whatever I get, you would get something different if the neighbors were different. (We would get the same if the neighbors were the same.) What is the chance that you get the same result as me? The chance is the number of times that the neighbor is the same color. If you'll think a minute, you'll find that two thirds of the time, in the case of Figure 5a, it's the same color. The worst case would be black/white/black/white/black/white, and there the probability of a match would be zero (Figure 5c,d). If you look at all eight possible distinct cases, you'll find that the biggest possible answer is two-thirds. You cannot arrange, in a classical kind of method like this, that the probability of agreement at 30° will be bigger than two-thirds. But the quantum mechanical formula predicts $\cos^2 30°$ (or 3/4)—and experiments agree with this—and therein lies the difficulty.

That's all. That's the difficulty. That's why quantum mechanics can't seem to be imitable by a local classical computer.

I've entertained myself always by squeezing the difficulty of quantum mechanics into a smaller and smaller place, so as to get more and more worried about this particular item. It seems to be almost ridiculous that you can squeeze it to a numerical question that one thing is bigger than another. But there you are—it is bigger than any logical argument can produce, if you have this kind of logic. Now, we say "this kind of logic;" what other possibilities are there? Perhaps there may be no possibilities, but perhaps there are. Its interesting to try to discuss the possibilities. I mentioned something about the possibility of time—of things being affected not just by the past, but also by the future, and therefore that our probabilities are in some sense "illusory." We only have the information from the past, and we try to predict the next step, but in reality it depends upon the near future which we can't get at, or something like that. A very interesting question is the origin of the probabilities in quantum mechanics. Another way of puttings things is this: we have an illusion that we can do any experiment that we want. We all, however, come from the same universe, have evolved with it, and don't really have any "real" freedom. For we obey certain laws and have come from a certain past. Is it somehow that we are correlated to the experiments that we do, so that the apparent probabilities don't look like they ought to look if you assume that they are random. There are all kinds of questions like this, and what I'm trying to do is to get you people who think about computer-simulation possibilities to pay a great deal of attention to this, to digest as well as possible the real answers of quantum mechanics, and see if you can't invent a different point of view than the physicists have had to invent to describe this. In fact the physicists have no

good point of view. Somebody mumbled something about a many-world picture, and that many-world picture says that the wave function ψ is what's real, and damn the torpedos if there are so many variables, N^R. All these different worlds and every arrangement of configurations are all there just like our arrangement of configurations, we just happen to be sitting in this one. It's possible, but I'm not very happy with it.

So, I would like to see if there's some other way out, and I want to emphasize, or bring the question here, because the discovery of computers and the thinking about computers has turned out to be extremely useful in many branches of human reasoning. For instance, we never really understood how lousy our understanding of languages was, the theory of grammar and all that stuff, until we tried to make a computer which would be able to understand language. We tried to learn a great deal about psychology by trying to understand how computers work. There are interesting philosophical questions about reasoning, and relationship, observation, and measurement and so on, which computers have stimulated us to think about anew, with new types of thinking. And all I was doing was hoping that the computer-type of thinking would give us some new ideas, if any are really needed. I don't know, maybe physics is absolutely OK the way it is. The program that Fredkin is always pushing, about trying to find a computer simulation of physics, seem to me to be an excellent program to follow out. He and I have had wonderful, intense, and interminable arguments, and my argument is always that the real use of it would be with quantum mechanics, and therefore full attention and acceptance of the quantum mechanical phenomena—the challenge of explaining quantum mechanical phenomena —has to be put into the argument, and therefore these phenomena have to be understood very well in analyzing the situation. And I'm not happy with all the analyses that go with just the classical theory, because nature isn't classical, dammit, and if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy. Thank you.

9. DISCUSSION

Question: Just to interpret, you spoke first of the probability of A given B, versus the probability of A and B jointly—that's the probability of one observer seeing the result, assigning a probability to the other; and then you brought up the paradox of the quantum mechanical result being 3/4, and this being 2/3. Are those really the same probabilities? Isn't one a joint probability, and the other a conditional one?

Answer: No, they are the same. P_{OO} is the *joint probability* that both you and I observe an ordinary ray, and P_{EE} is the *joint* probability for two

extraordinary rays. The probability that our observations match is

$$P_{OO} + P_{EE} = \cos^2 30° = 3/4$$

Question: Does it in some sense depend upon an assumption as to how much information is accessible from the photon, or from the particle? And second, to take your question of prediction, your comment about predicting, is in some sense reminiscent of the philosophical question, Is there any meaning to the question of whether there is free will or predestination? namely, the correlation between the observer and the experiment, and the question there is, Is it possible to construct a test in which the prediction could be reported to the observer, or instead, has the ability to represent information already been used up? And I suspect that you may have already used up all the information so that prediction lies outside the range of the theory.

Answer: All these things I don't understand; deep questions, profound questions. However physicists have a kind of a dopy way of avoiding all of these things. They simply say, now look, friend, you take a pair of counters and you put them on the side of your calcite and you count how many times you get this stuff, and it comes out 75% of the time. Then you go and you say, Now can I imitate that with a device which is going to produce the same results, and which will operate locally, and you try to invent some kind of way of doing that, and if you do it in the ordinary way of thinking, you find that you can't get there with the same probability. Therefore some new kind of thinking is necessary, but physicists, being kind of dull minded, only look at nature, and don't know how to think in these new ways.

Question: At the beginning of your talk, you talked about discretizing various things in order to go about doing a real computation of physics. And yet it seems to me that there are some differences between things like space and time, and probability that might exist at some place, or energy, or some field value. Do you see any reason to distinguish between quantization or discretizing of space and time, versus discretizing any of the specific parameters or values that might exist?

Answer: I would like to make a few comments. You said quantizing or discretizing. That's very dangerous. Quantum theory and quantizing is a very specific type of theory. Discretizing is the right word. Quantizing is a different kind of mathematics. If we talk about discretizing... of course I pointed out that we're going to have to change the laws of physics. Because the laws of physics as written now have, in the classical limit, a continuous variable everywhere, space and time. If, for example, in your theory you were going to have an electric field, then the electric field could not have (if it's going to be imitable, computable by a finite number of elements) an

infinite number of possible values, it'd have to be digitized. You might be able to get away with a theory by redescribing things without an electric field, but supposing for a moment that you've discovered that you can't do that and you want to describe it with an electric field, then you would have to say that, for example, when fields are smaller than a certain amount, they aren't there at all, or something. And those are very interesting problems, but unfortunately they're not good problems for classical physics because if you take the example of a star a hundred light years away, and it makes a wave which comes to us, and it gets weaker, and weaker, and weaker, and weaker, the electric field's going down, down, down, how low can we measure? You put a counter out there and you find "clunk," and nothing happens for a while, "clunk," and nothing happens for a while. It's not discretized at all, you never can measure such a tiny field, you don't find a tiny field, you don't have to imitate such a tiny field, because the world that you're trying to imitate, the physical world, is not the classical world, and it behaves differently. So the particular example of discretizing the electric field, is a problem which I would not see, as a physicist, as fundamentally difficult, because it will just mean that your field has gotten so small that I had better be using quantum mechanics anyway, and so you've got the wrong equations, and so you did the wrong problem! That's how I would answer that. Because you see, if you would imagine that the electric field is coming out of some 'ones' or something, the lowest you could get would be a full one, but that's what we see, you get a full photon. All these things suggest that it's really true, somehow, that the physical world is representable in a discretized way, because every time you get into a bind like this, you discover that the experiment does just what's necessary to escape the trouble that would come if the electric field went to zero, or you'd never be able to see a star beyond a certain distance, because the field would have gotten below the number of digits that your world can carry.

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Quantum Mechanical Computers[1]

Richard P. Feynman[2]

Received March 15, 1985

The physical limitations, due to quantum mechanics, on the functioning of computers are analyzed.

1. INTRODUCTION

This work is a part of an effort to analyze the physical limitations of computers due to the laws of physics. For example, Bennett[(1)] has made a careful study of the free energy dissipation that must accompany computation. He found it to be virtually zero. He suggested to me the question of the limitations due to quantum mechanics and the uncertainty principle. I have found that, aside from the obvious limitation to size if the working parts are to be made of atoms, there is no fundamental limit from these sources either.

We are here considering ideal machines; the effects of small imperfections will be considered later. This study is one of principle; our aim is to exhibit some Hamiltonian for a system which could serve as a computer. We are not concerned with whether we have the most efficient system, nor how we could best implement it.

Since the laws of quantum physics are reversible in time, we shall have to consider computing engines which obey such reversible laws. This problem already occurred to Bennett,[(1)] and to Fredkin and Toffoli,[(2)] and a great deal of thought has been given to it. Since it may not be familiar to

[1] *Editor's note*: This article, which is based on the author's plenary talk presented at the CLEO/IQEC Meeting in 1984, originally appeared in the February 1985 issue of *Optics News*. It is here reprinted with kind permission of Professor Feynman and *Optics News*.

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you here, I shall review this, and in doing so, take the opportunity to review, very briefly, the conclusions of Bennett,[2] for we shall confirm them all when we analyze our quantum system.

It is a result of computer science that a universal computer can be made by a suitably complex network of interconnected primitive elements. Following the usual classical analysis we can imagine the interconnections to be ideal wires carrying one of two standard voltages representing the local 1 and 0. We can take the primitive elements to be just two, NOT and AND (actually just the one element NAND = NOT AND suffices, for if one input is set at 1 the output is the NOT of the other input). They are symbolized in Fig. 1, with the logical values resulting on the outgoing wires, resulting from different combinations of input wires.

From a logical point of view, we must consider the wires in detail, for in other systems, and our quantum system in particular, we may not have wires as such. We see we really have two more logical primitives, FAN OUT when two wires are connected to one, and EXCHANGE, when wires are crossed. In the usual computer the NOT and NAND primitives are implemented by transistors, possibly as in Fig. 2.

What is the minimum free energy that must be expended to operate an ideal computer made of such primitives? Since, for example, when the AND operates the output line, c' is being determined to be one of two values, no matter what it was before, the entropy change is ln 2 units. This represents a heat generation of kT ln 2 at temperature T. For many years it was thought that this represented an absolute minimum to the quantity of heat per primitive step that had to be dissipated in making a calculation.

The question is academic at this time. In actual machines we are quite concerned with the heat dissipation question, but the transistor system used actually dissipates about $10^{10}kT$! As Bennett[3] has pointed out, this arises because to change a wire's voltage we dump it to ground through a resistance; and to build it up again we feed charge, again through a resistance, to the wire. It could be greatly reduced if energy could be stored in an inductance, or other reactive element.

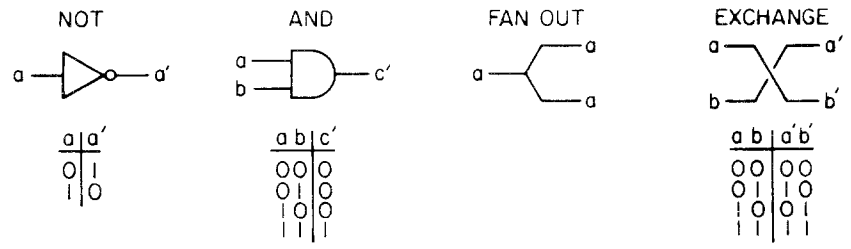


Fig. 1. Primitive elements.

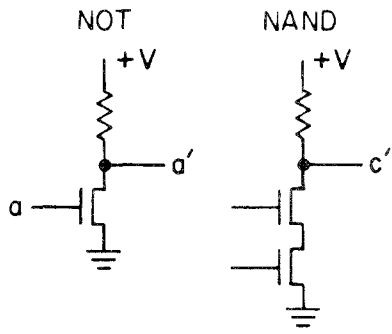


Fig. 2. Transistor circuits for NOT and NAND.

However, it is apparently very difficult to make inductive elements on silicon wafers with present techniques. Even Nature, in her DNA copying machine, dissipates about $100kT$ per bit copied. Being, at present, so very far from this $kT \ln 2$ figure, it seems ridiculous to argue that even this is too high and the minimum is really essentially zero. But, we are going to be even more ridiculous later and consider bits written on one atom instead of the present 10^{11} atoms. Such nonsense is very entertaining to professors like me. I hope you will find it interesting and entertaining also.

What Bennett pointed out was that this former limit was wrong because it is not necessary to use irreversible primitives. Calculations can be done with reversible machines containing only reversible primitives. If this is done the minimum free energy required is independent of the complexity or number of logical steps in the calculation. If anything, it is kT per bit of the output answer.

But even this, which might be considered the free energy needed to clear the computer for further use, might also be considered as part of what you are going to do with the answer—the information in the result if you transmit it to another point. This is a limit only achieved ideally if you compute with a reversible computer at infinitesimal speed.

2. COMPUTATION WITH A REVERSIBLE MACHINE

We will now describe three reversible primitives that could be used to make a universal machine (Toffoli[4]). The first is the NOT which evidently loses no information, and is reversible, being reversed by acting again with NOT. Because the conventional symbol is not symmetrical we shall use an X on the wire instead (see Fig. 3a).

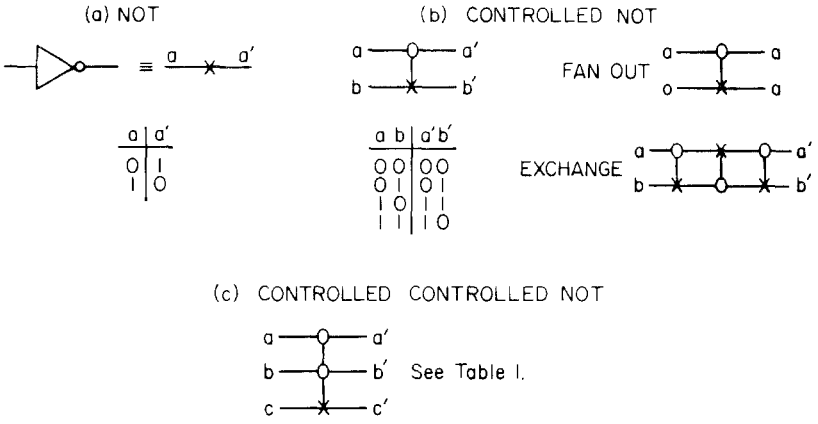


Fig. 3. Reversible primitives.

Next is what we shall call the CONTROLLED NOT (see Fig. 3b). There are two entering lines, a and b, and two exiting lines, a' and b'. The a' is always the same as a, which is the control line. If the control is activated $a = 1$ then the out b' is the NOT of b. Otherwise b is unchanged, $b' = b$. The table of values for input and output is given in Fig. 3. The action is reversed by simply repeating it.

The quantity b' is really a symmetric function of a and b called XOR, the exclusive or; a or b but not both. It is likewise the sum modulo 2 of a and b, and can be used to compare a and b, giving a 1 as a signal that they are different. Please notice that this function XOR is itself not reversible. For example, if the value is zero we cannot tell whether it came from $(a, b) = (0, 0)$ or from $(1, 1)$ but we keep the other line $a' = a$ to resolve the ambiguity.

We will represent the CONTROLLED NOT by putting a 0 on the control wire, connected with a vertical line to an X on the wire which is controlled.

This element can also supply us with FAN OUT, for if $b = 0$ we see that a is copied onto line b'. This COPY function will be important later on. It also supplies us with EXCHANGE, for three of them used

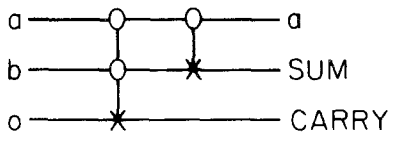


Fig. 4. Adder.

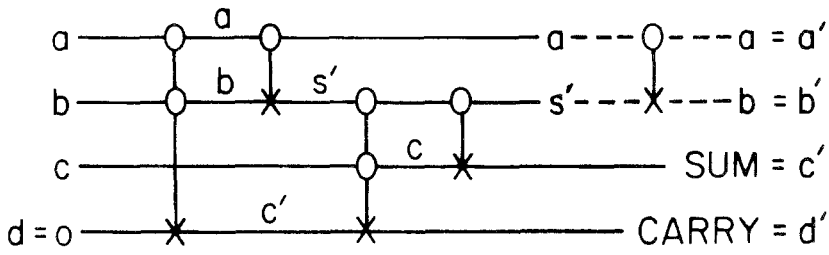


Fig. 5. Full adder.

successively on a pair of lines, but with alternate choice for control line, accomplishes an exchange of the information on the lines (Fig. 3b).

It turns out that combinations of just these two elements alone are insufficient to accomplish arbitrary logical functions. Some element involving three lines is necessary. We have chosen what we can call the CONTROLLED CONTROLLED NOT. Here (see Fig. 3c) we have two control lines a, b, which appear unchanged in the output and which change the third line c to NOT c only if both lines are activated ($a = 1$ and $b = 1$). Otherwise $c' = c$. If the third line input c is set to 0, then evidently it becomes $1 (c' = 1)$ only if both a and b are 1 and therefore supplies us with the AND function (see Table I).

Three combinations for (a, b), namely (0, 0), (0, 1), and (1, 0), all give the same value, 0, to the AND (a, b) function so the ambiguity requires two bits to resolve it. These are kept in the lines a, b in the output so the function can be reversed (by itself, in fact). The AND function is the carry bit for the sum of a and b.

From these elements it is known that any logical circuit can be put together by using them in combination, and in fact, computer science

Table I.

a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

shows that a universal computer can be made. We will illustrate this by a little example. First, of course, as you see in Fig. 4, we can make an adder, by first using the CONTROLLED CONTROLLED NOT and then the CONTROLLED NOT in succession, to produce from a and b and 0, as input lines, the original a on one line, the sum on the second line, and the carry on the third.

A more elaborate circuit is a full adder (see Fig. 5), which takes a carry c (from some previous addition) and adds it to the two lines a and b and has an additional line d with a 0 input. It requires four primitive elements to be put together. Besides this total sum, the total of the three, a, b, and c and the carry, we obtain on the other two lines two pieces of information. One is the a that we started with, and the other is some intermediary quantity that we calculated on route.

This is typical of these reversible systems; they produce not only what you want in output, but also a certain amount of garbage. In this particular case, and as it turns out in all cases, the garbage can be arranged to be, in fact, just the input, if we would just add the extra CONTROLLED NOT on the first two lines, as indicated by the dotted lines in Fig. 5; we see that the garbage would become a and b, which were the inputs of at least two of the lines. (We know this circuit can be simplified but we do it this way for illustrative purposes.)

In this way, we can by various combinations produce a general logic unit that transforms n bits to n bits in a reversible manner. If the problem you are trying to do is itself reversible, then there might be no extra garbage, but in general, there are some extra lines needed to store up the information which you would need to be able to reverse the operation. In other words, we can make any function that the conventional system can, plus garbage. The garbage contains the information you need to reverse the process.

And how much garbage? It turns out in general that if the output data that you are looking for has k bits, then starting with an input and k bits containing 0, we can produce, as a result, just the input and the output and no further garbage. This is reversible because knowing the output and the input permits you, of course, to undo everything. This proposition is always reversible. The argument for this is illustrated in Fig. 6.

Suppose we began with a machine M, which, starting with an input, and some large number of 0's, produces the desired outut plus a certain amount of extra data which we call garbage. Now we have seen that the copy operation which can be done by a sequence of CONTROLLED NOT's is possible, so if we have originally an empty register, with the k bits ready for the output, we can, after the processor M has operated, copy the output from the M onto this new register.

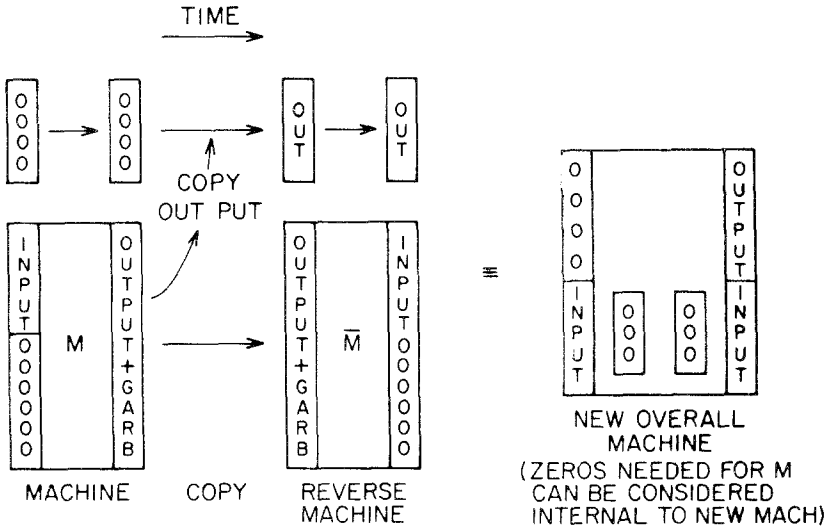


Fig. 6. Clearing garbage.

After that, we can build the opposite machine, the M in reverse, the reverse machine, which would take this output of M and garbage and turn it into the input and 0's. Thus, seen as an overall machine, we would have started with the k 0's of the register for the output, and the input, and ended up with those k 0's occupied by the output data, and repeat the inut as a final product. The number of 0's that was originally needed in the M machine in order to hold the garbage is restored again to 0, and can be considered as internal wires inside the new complete machine (M, $\bar{M}$ and copy).

Overall, then, we have accomplished what we set out to do, and therefore garbage need never be any greater than a repetition of the input data.

3. A QUANTUM MECHANICAL COMPUTER

We now go on to consider how such a computer can also be built using the laws of quantum mechanics. We are going to write a Hamiltonian, for a system of interacting parts, which will behave in the same way as a large system in serving as a universal computer. Of course the large system also obeys quantum mechanics, but it is in interaction with the heat baths and other things that could make it effectively irreversible.

What we would like to do is make the computer as small and as

simple as possible. Our Hamiltonian will describe in detail all the internal computing actions, but not, of course, those interactions with the exterior involved in entering the input (preparing the initial state) and reading the output.

How small can such a computer be? How small, for instance, can a number be? Of course a number can be represented by bits of 1's and 0's. What we are going to do is imagine that we have two-state systems, which we will call "atoms." An n bit number is then represented by a state of a "register," a set of n two-state systems.

Depending upon whether or not each atom is in one or another of its two states, which we call $|1\rangle$ and $|0\rangle$, we can of course, represent any number. And the number can be read out of such a register by determining, or measuring, in which state each of the atoms are at a given moment. Therefore one bit will be represented by a single atom being in one of two states, the states we will call $|1\rangle$ and $|0\rangle$.

What we will have to do then can be understood by considering an example; the example of a CONTROLLED CONTROLLED NOT. Let G be some sort of an operation on three atoms a, b, and c, which converts the original state of a, b, and c into a nex appropriate state, a', b', c', so that the connection between a', b', and c' and a, b, c, are just what we would have expected if a, b, and c represented wires, and the a', b', and c' were the output wires of a CONTROLLED CONTROLLED NOT.

It must be appreciated here that at the moment we are not trying to move the data from one position to another; we are just going to change it. Unlike the situation in the actual wired computer in which the voltages on one wire then go over to voltages on another, what we are specifically making is something simpler, that the three atoms are in some particular state, and that an operation is performed, which changes the state to new values, a', b', c'.

What we would have then is that the state, in the mathematical form $|a', b', c'\rangle$, is simply some operation G operating on $|a, b, c\rangle$. In quantum mechanics, state changing operators are linear operators, and so we'll suppose that G is linear. Therefore, G is a matrix, and the matrix elements of G, $G_{a',b',c',a,b,c}$ are all 0 except those in Table I, which are of course 1.

This table is the same table that represents the truth value table for the CONTROLLED CONTROLLED NOT. It is apparent that the operation is reversible, and that can be represented by saying that $G^*G = 1$, where the * means Hermitian adjoint. That is to say, G is a unitary matrix. (In fact G is also a real matrix $G^* = G$, but that's only a special case.) To be more specific, we are going to write $A_{ab,c}$ for this special G. We shall use the same matrix A with different numbers of subscripts to represent the other primitive elements.

To take a simple example, the NOT, which would be represented by A_a, is the simple matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This is a 2×2 matrix and can be represented in many ways, in different notations, but the particular one we will use to define these is by the method of creation and annihilation operators. Consider operating in this case, on a single line a. In order to save alphabets, let us call $\underline{a}$ the matrix

$$\underline{a} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which annihilates the 1 on atom a and converts it to 0; $\underline{a}$ is an operator which converts the state $|1\rangle$ to $|0\rangle$. But, if the state of the atom were originally $|0\rangle$, the operator $\underline{a}$ produces the number 0. That is, it doesn't change the state, it simply produces the numerical value zero when operating on that state. The conjugate of this thing, of course, is

$$\underline{a^*} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

which creates, in the sense that operating on the 0 state, it turns it to the 1 state. In other words, it moves from $|0\rangle$ to $|1\rangle$. When operating on the $|1\rangle$ state there is no further state above that which you can create, and therefore it gives it the number zero. Every other operator 2×2 matrix can be represented in terms of these $\underline{a}$ and $\underline{a^*}$. For example, the product $\underline{a^*}\underline{a}$ is equal to the matrix

$$\underline{a^* a} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

which you might call N_a. It is 1 when the state is $|1\rangle$ and 0 when the state is $|0\rangle$. It gives the number that the state of the atom represents. Likewise the product

$$\underline{a a^*} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is $1 - N_a$, and gives 0 for the up-state and 1 for the down-state. We'll use 1 to represent the diagonal matrix,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As a consequence of all this, $\underline{a a^*} + \underline{a^* a} = 1$.

It is evident then that our matrix for NOT, the operator that produces NOT, is $A_a = \underline{a} + \underline{a}^*$ and further of course, that's reversible, $A_a{}^* A_a = 1$, A_a is unitary.

In the same way the matrix $A_{a,b}$ for the CONTROLLED NOT can be worked out. If you look at the table of values for CONTROLLED NOT you see that it can be written this way:

$$\underline{a}^* \underline{a}(\underline{b} + \underline{b}^*) + \underline{a}\underline{a}^*$$

In the first term, the $\underline{a}^* \underline{a}$ selects the condition that the line $a = 1$, in which case we want $\underline{b} + \underline{b}^*$ the NOT to apply to b. The second term selects the condition that the line a is 0, in which case we want nothing to happen to b and the unit matrix on the operators of b is implied. This can also be written as $1 + \underline{a}^* \underline{a}(\underline{b} + \underline{b}^* - 1)$, the 1 representing all the lines coming through directly, but in the case that a is 1, we would like to correct that by putting in a NOT instead of leaving the line b unchanged.

The matrix for the CONTROLLED CONTROLLED NOT is

$$A_{ab,c} = 1 + \underline{a}^* \underline{a}\underline{b}^* \underline{b}(\underline{c} + \underline{c}^* - 1)$$

as, perhaps, you may be able to see.

The next question is what the matrix is for a general logic unit which consists of a sequence of these. As an example, we'll study the case of the full adder which we described before (see Fig. 5). Now we'll have, in the general case, four wires represented by a, b, c, and d; we don't necessarily have to have d as 0 in all cases, and we would like to describe how the object operates in general (if d is changed to 1 d' is changed to its NOT). It produces new numbers a', b', c', and d', and we could imagine with our system that there are four atoms labeled a, b, c, d in a state labeled $|a, b, c, d\rangle$ and that a matrix M operates which changes these same four atoms so that they appear to be in the state $|a', b', c', d'\rangle$ which is appropriate for this logic unit. That is, if $|\psi_{\text{in}}\rangle$ represents the incoming state of the four bits, M is a matrix which generates an outgoing state $|\psi_{\text{out}}\rangle = M|\psi_{\text{in}}\rangle$ for the four bits.

For example, if the input state were the state $|1, 0, 1, 0\rangle$, then, as we know, the output state should be $|1, 0, 0, 1\rangle$; the first two a', b' should be 1, 0 for those two first lines come streight through, and the last two c', d' should be 0, 1 because that represents the sum and carry of the first three, a, b, c, bits in the first input, as $d = 0$. Now the matrix M for the adder can easily be seen as the result of five successive primitive operations, and therefore becomes the matrix product of the five successive matrices representing these primitive objects.

$$M = A_{a,b} A_{b,c} A_{bc,d} A_{a,b} A_{ab,d}$$

The first, which is the one written farthest to the right, is $A_{ab,d}$ for that represents the CONTROLLED CONTROLLED NOT in which a and b are the CONTROL lines, and the NOT appears on line d. By looking at the diagram in Fig. 5 we can immediately see what the remaining factors in the sequence represent. The last factor, for example, $A_{a,b}$, means that there's a CONTROLLED NOT with a CONTROL on line a and NOT on line b. This matrix will have the unitary property $M^*M = 1$ since all of the A's out of which it is a product are unitary. That is to say, M is a reversal operation, and M^* is its inverse.

Our general problem, then, is this. Let $A_1, A_2, A_3, \ldots, A_k$ be the succession of operations wanted, in some logical unit, to operate on n lines. The $2'' \times 2''$ matrix M needed to accomplish the same goal is a product $A_k \cdots A_3 A_2 A_1$, where each A is a simple matrix. How can we generate this M in a physical way if we know how to make the simpler elements?

In general, in quantum mechanics, the outgoing state at time t is $e^{iHt}\psi_{\text{in}}$, where ψ_{in} is the input state, for a system with Hamiltonian H. To try to find, for a given special time t, the Hamiltonian which will produce $M = e^{iHt}$ when M is such a product of noncommuting matrices, from some simple property of the matrices themselves, appears to be very difficult.

We realize however, that at any particular time, if we expand the e^{iHt} out (as $1 + iHt - H^2t^2/2 - \cdots$) we'll find the operator H operating an innumerable arbitrary number of times, once, twice, three times, and so forth, and the total state is generated by a superposition of these possibilities. This suggests that we can solve this problem of the composition of these A's in the following way.

We add to the n atoms, which are in our register, an entirely new set of $k+1$ atoms, which we'll call "program counter sites." Let us call q_i and q_i^* the annihilation and creation operators for the program site i for $i = 0$ to k. A good thing to think of, as an example, is an electron moving from one empty site to another. If the site i is occupied by the electron, its state is $|1\rangle$, while if the site is empty, its state is $|0\rangle$.

We write, as our Hamiltonian

$$H = \sum_{i=0}^{k-1} q_{i+1}^* q_i A_{i+1} + \text{complex conjugate}$$

$$= q_1^* q_0 A_1 + q_2^* q_1 A_2 + q_3^* q_2 A_3 + \cdots + q_0^* q_1 A_1^*$$

$$+ q_1^* q_2 A_2^* + q_2^* q_3 A_3^* + \cdots$$

The first thing to notice is that if all the program sites are unoccupied, that is, all the program atoms are initially in the state 0, nothing happens

because every term in the Hamiltonian starts with an annihilation operator and it gives 0 therefore.

The second thing we notice is that if only one or another of the program sites is occupied (in state $|1\rangle$), and the rest are not (state $|0\rangle$), then this is always true. In fact the number of program sites that are in state $|1\rangle$ is a conserved quantity. We will suppose that in the operation of this computer, either no sites are occupied (in which case nothing happens) or just one site is occupied. Two or more program sites are never both occupied during normal operation.

Let us start with an initial state where site 0 is occupied, is in the $|1\rangle$ state, and all the others are empty, $|0\rangle$ state. If later, at some time, the final site k is found to be in the $|1\rangle$ state, (and therefore all the others in $|0\rangle$) then, we claim, the n register has been multiplied by the matrix M, which is $A_k \cdots A_2 A_1$ as desired.

Let me explain how this works. Suppose that the register starts in any initial state, ψ_{in}, and that the site, 0, of the program counter is occupied. Then the only term in the entire Hamiltonian that can first operate, as the Hamiltonian operates in successive times, is the first term, $q_1^* q_0 A_1$. The q_0 will change site number 0 to an unoccupied site, while q_1^* will change the site number 0 to an occupied site. Thus the term $q_1^* q_0$ is a term which simply moves the occupied site from the location 0 to the location 1. But this is multiplied by the matrix A_1 which operates only on the n register atoms, and therefore multiplies the initial state of the n register atoms by A_1.

Now, if the Hamiltonian happens to operate a second time, this first term will produce nothing because q_0 produces 0 on the number 0 site because it is now unoccupied. The term which can operate now is the second term, $q_2^* q_1 A_2$, for that can move the occupied point, which I shall call a "cursor." The cursor can move from site 1 to site 2 but the matrix A_2 now operates on the register; therefore the register has now got the matrix $A_2 A_1$ operating on it.

So, looking at the first line of the Hamiltonian, if that is all there was to it, as the Hamiltonian operates in successive orders, the cursor would move successively from 0 to k, and you would acquire, one after the other, operating on the n register atoms, the matrices, A, in the order that we would like to construct the total M.

However, a Hamiltonian must be hermitian, and therefore the complex conjugate of all these operators must be present. Suppose that at a given stage, we have gotten the cursor on site number 2, and we have the matrix $A_2 A_1$ operating on the register. Now the q_2 which intends to move that occupation to a new position need not come from the first line, but may have come from the second line. It may have come, in fact, from

$q_1^* q_2 A_2^*$ which would move the cursor back from the position 2 to the position 1.

But note that when this happens, the operator A_2^* operates on the register, and therefore the total operator on the register is $A_2^* A_2 A_1$ in this case. But $A_2^* A_2$ is 1 and therefore the operator is just A_1. Thus we see that when the cursor is returned to the position 1, the net result is that only the operator A_1 has really operated on the register. Thus it is that as the various terms of the Hamiltonian move the cursor forwards and backwards, the A's accumulate, or are reduced out again.

At any stage, for example, if the cursor were up to the j site, the matrices from A_1 to A_j have operated in succession on the n register. It does not matter whether or not the cursor on the j site has arrived there, by going directly from 0 to j, or going further and returning, or going back and forth in any pattern whatsoever, as long as it finally arrived at the state j.

Therefore it is true that if the cursor is found at the site k, we have the net result for the n register atoms that the matrix M has operated on their initial state as we desired.

How then could we operate this computer? We begin by putting the input bits onto the register, and by putting the cursor to occupy the site 0. We then check at the site k, say, by scattering electrons, that the site k is empty, or that the site k has a cursor. The moment we find the cursor at site k we remove the cursor so that it cannot return down the program line, and then we know that the register contains the output data. We can then measure it at our leisure. Of course, there are external things involved in making the measurements, and determining all of this, which are not part of our computer. Surely a computer has eventually to be in interaction with the external world, both for putting data in and for taking it out.

Mathematically it turns out that the propagation of the cursor up and down this program line is exactly the same as it would be if the operators A were not in the Hamiltonian. In other words, it represents just the waves which are familiar from the propagation of the tight binding electrons or spin waves in one dimension, and are very well known. There are waves that travel up and down the line and you can have packets of waves and so forth.

We could improve the action of this computer and make it into a ballistic action in the following way: by making a line of sites in addition to the ones inside, that we are actually using for computing, a line of say, many sites, both before and after. It's just as though we had values of the index i for q_i, which are less than 0 and greater than k, each of which has no matrix A, just a 1 multiplying there. Then we had have a longer spin chain, and we could have started, instead of putting a cursor exactly at the

beginning site 0, by putting the cursor with different amplitudes on different sites representing an initial incoming spin wave, a wide packet of nearly definite momentum.

This spin wave would then go through the entire computer in a ballistic fashion and out the other end into the outside tail that we have added to the line of program sites, and there it would be easier to determine if it is present and to steer it away to some other place, and to capture the cursor. Thus, the logical unit can act in a ballistic way.

This is the essential point and indicates, at least to a computer scientist, that we could make a universal computer, because he knows if we can make any logical unit we can make a universal computer. That this could represent a universal computer for which composition of elements and branching can be done is not entirely obvious unless you have some experience, but I will discuss that to some further extent later.

4. IMPERFECTIONS AND IRREVERSIBLE FREE ENERGY LOSS

There are, however, a number of questions that we would like to discuss in more detail such as the question of imperfections.

There are many sources of imperfections in this machine, but the first one we would like to consider is the possibility that the coefficients in the couplings, along the program line, are not exactly equal. The line is so long that in a real calculation little irregularities would produce a small probability of scattering, and the waves would not travel exactly ballistically, but would go back and forth.

If the system, for example, is built so that these sites are built on a substrate of ordinary physical atoms, then the thermal vibrations of these atoms would change the couplings a little bit and generate imperfections. (We should even need such noise for with small fixed imperfections there are shallow trapping regions where the cursor may get caught.) Suppose then, that there is a certain probability, say p per step of calculation (that is, per step of cursor motion, $i \to i+1$), for scattering the cursor momentum until it is randomized ($1/p$ is the transport mean free path). We will suppose that the p is fairly small.

Then in a very long calculation, it might take a very long time for the wave to make its way out the other end, once started at the beginning —because it has to go back and forth so many times due to the scattering. What one then could do would be to pull the cursor along the program line with an external force. If the cursor is, for example, an electron moving from one vacant site to another, this would be just like an electric field trying to pull the electron along a wire, the resistance of which is generated

by the imperfection or the probability of scattering. Under these circumstances we can calculate how much energy will be expended by this external force.

This analysis can be made very simply: it is an almost classical analysis of an electron with a mean free path. Every time the cursor is scattered, I am going to suppose it is randomly scattered forward and backward. In order for the machine to operate, of course, it must be moving forward at a higher probability than it is moving backward. When a scattering occurs therefore, the loss in entropy is the logarithm of the probability that the cursor is moving forward, divided by the probability the cursor was moving backward.

This can be approximated by (the probability forward − the probability backward)/(the probability forward + the probability backward). That was the entropy lost per scattering. More interesting is the entropy lost per net calculational step, which is, of course, simply p times that number. We can rewrite the entropy cost per calculational step as

$$pv_D/v_R$$

where v_D is the drift velocity of the cursor and v_R its random velocity.

Or if you like, it is p times the minimum time that the calculation could be done in (that is, if all the steps were always in the forward direction), divided by the actual time allowed.

The free energy loss per step then, is $kT \times p \times$ the minimum time that the calculation could be done, divided by the actual time that you allow yourself to do it. This is a formula that was first derived by Bennett. The factor p is a coasting factor, to represent situations in which not every site scatters the cursor randomly, but it has only a small probability to be thus scattered.

It will be appreciated that the energy loss per step is not kT but is that divided by two factors. One, $(1/p)$, measures how perfectly you can build the machine and the other is proportional to the length of time that you take to do the calculation. It is very much like a Carnot engine, in which in order to obtain reversibility, one must operate very slowly. For the ideal machine where p is 0, or where you allow an infinite time, the mean energy loss can be 0.

The uncertainty principle, which usually relates some energy and time uncertainty, is not directly a limitation. What we have in our computer is a device for making a computation, but the time of arrival of the cursor and the measurement of the output register at the other end (in other words, the time it takes in which to complete the calculation) is not a define time. It's a question of probabilities, and so there is a considerable uncertainty in the time at which a calculation will be done.

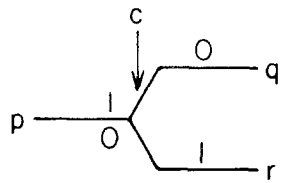
There is no loss associated with the uncertainty of cursor energy; at least no loss depending on the number of calculational steps. Of course, if you want to do a ballistic calculation on a perfect machine, some energy would have to be put into the original wave, but that energy, of course, can be removed from the final wave when it comes out of the tail of the program line. All questions associated with the uncertainty of operators and the irreversibility of measurements are associated with the input and output functions.

No further limitations are generated by the quantum nature of the computer *per se*, nothing that is proportional to the number of computational steps.

In a machine such as this, there are very many other problems, due to imperfections. For example, in the registers for holding the data, there will be problems of cross-talk, interactions between one atom and another in that register, or interaction of the atoms in that register directly with things that are happening along the program line, that we did not exactly bargain for. In other words, there may be small terms in the Hamiltonian besides the ones we have written.

Until we propose a complete implementation of this, it is very difficult to analyze. At least some of these problems can be remedied in the usual way by techniques such as error correcting codes, and so forth, that have been studied in normal computers. But until we find a specific implementation for this computer, I do not know how to proceed to analyze these effects. However, it appears that they would be very important, in practice. This computer seems to be very delicate and these imperfections may produce considerable havoc.

The time needed to make a step of calculation depends on the strength or the energy of the interactions in the terms of the Hamiltonian. If each of the terms in the Hamiltonian is supposed to be of the order of 0.1 electron volts, then it appears that the time for the cursor to make each step, if done in a ballistic fashion, is of the order 6×10^{-15} sec. This does not represent



$H = q^* cp + r^* c^* p + p^* c^* q + p^* cr$

IF c = 1 GO p TO q AND PUT c = 0
IF c = 0 GO p TO r AND PUT c = 1
IF c = 1 GO r TO p AND PUT c = 0
IF c = 0 GO q TO p AND PUT c = 1

Fig. 7. Switch.

an enormous improvement, perhaps only about four orders of magnitude over the present values of the time delays in transistors, and is not much shorter than the very short times possible to achieve in many optical systems.

5. SIMPLIFYING THE IMPLEMENTATION

We have completed the job we set out to do—to find some quantum mechanical Hamiltonian of a system that could compute, and that is all that we need say. But it is of some interest to deal with some questions about simplifying the implementation. The Hamiltonian that we have written involves terms which can involve a special kind of interaction between five atoms. For example, three of them in the register, for a CONTROLLED CONTROLLED NOT, and two of them as the two adjacent sites in the program counter.

This may be rather complicated to arrange. The question is, can we do it with simpler parts. It turns out that we can indeed. We can do it so that in each interaction there are only three atoms. We are going to start with new primitive elements, instead of the ones we began with. We'll have the NOT all right, but we have in addition to that simply a "switch" (see also Priese[5]).

Supposing that we have a term, $q^*cp + r^*c^*p +$ its complex conjugate in the Hamiltonian (in all cases we'll use letters in the earlier part of the alphabet for register atoms and in the latter part of the alphabet for program sites). See Fig. 7. This is a switch in the sense that, if c is originally in the $|1\rangle$ state, a cursor at p will move to q, whereas if c is in the $|0\rangle$ state, the cursor at p will move to r.

During this operation the controlling atom c changes its state. (It is possible also to write an expression in which the control atom does not change its state, such as $q^*c^*cp + r^*cc^*p$ and its complex conjugate but, there is no particular advantage or disadvantage to this, and we will take the simpler form.) The complex conjugate reverses this.

If, however, the cursor is at q and c is in the state $|1\rangle$ (or cursor at r, c in $|0\rangle$), the H gives 0, and the cursor gets reflected back. We shall build all our circuits and choose initial states so that this circumstance will not arise in normal operation, and the ideal ballistic mode will work.

With this switch we can do a number of things. For example, we could produce a CONTROLLED NOT as in Fig. 8. The switch a controls the operation. Assume the cursor starts at s. If $a = 1$ the program cursor is carried along the top line, whereas if $a = 0$ it is carried along the bottom line, in either case terminating finally in the program site t.

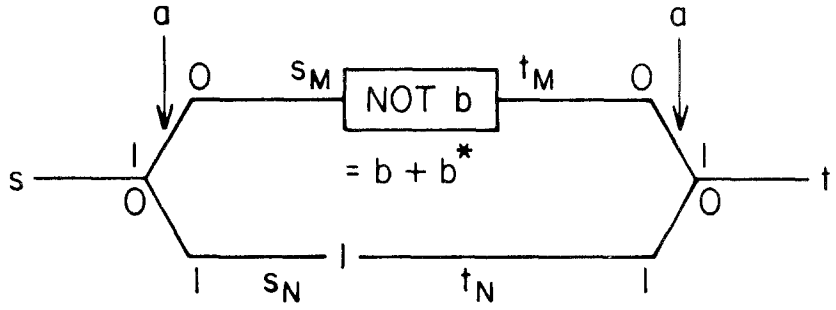


Fig. 8. CONTROLLED NOT by switches.

In these diagrams, horizontal or vertical lines will represent program atoms. The switches are represented by diagonal lines and in boxes we'll put the other matrices that operate on registers such as the NOT b. To be specific, the Hamiltonian for this little section of a CONTROLLED NOT, thinking of it as starting at s and ending at t, is given below:

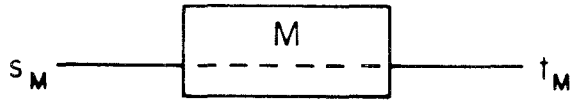
$$H_c(s,t) = s_M^* a s + t^* a^* t_M + t_M^*(b+b^*)\, s_M + s_N^* a^* s + t^* a t_N + t_N^* s_N + \text{c.c}$$

(The c.c means to add the complex conjugate of all the previous terms.)

Although there seem to be two routes here which would possibly produce all kinds of complications characteristic of quantum mechanics, this is not so. If the entire computer system is started in a definite state for a by the time the cursor reaches s, the atom a is still in some definite state (although possibly different from its initial state due to previous computer operations on it). Thus only one of the two routes is taken. The expression may be simplified by omitting the $s_N^* t_N$ term and putting $t_N = s_N$.

One need not be concerned in that case, that one route is longer (two cursor sites) than the other (one cursor site) for again there is no interference. No scattering is produced in any case by the insertion into a chain of coupled sites, an extra piece of chain of any number of sites with the same mutual coupling between sites (analogous to matching impedances in transmission lines).

To study these things further, we think of putting pieces together. A piece (see Fig. 9) M might be represented as a logical unit of interacting parts in which we only represent the first input cursor site as s_M and the final one at the other end as t_M. All the rest of the program sites that are between s_M and t_M are considered internal parts of M, and M contains its registers. Only s_M and t_M are sites that may be coupled externally.



s_M = Starting program site for piece

t_M = Terminal program site for piece

$H_M(s_M, t_M)$ is the part of the Hamiltonian representing all the "atoms" and program sites within the box M, and their interactions with s_M, t_M.

Fig. 9. One "piece."

The Hamiltonian for this subsection we'll call H_M and we'll identify s_M and t_M, as the name of the input and output program sites by writing $H_M(s_M, t_M)$. So therefore H_M is that part of the Hamiltonian representing all the atoms in the box and their external start and terminator sites.

An especially important and interesting case to consider is when the input data (in the regular atoms) comes from one logical unit, and we would like to transfer it to another (see Fig. 10). Suppose that we imagine that the box M starts with its input register with 0 and its output (which may be the same register) also with 0. Then we could use it in the following way. We could make a program line, let's say starting with s'_M whose first job is to exchange the data in an external register which contains the input, with M's input register which at the present time contains 0's.

Then the first step in our calculation, starting, say, at s'_M, would be to make an exchange with the register inside of M. That puts zero's into the original input register and puts the input where it belongs inside the box M. The cursor is now at s_M. (We have already explained how exchange can be made of controlled NOTs.) Then as the program goes from s_M to t_M we find the output now in the box M. Then the output register of M is now cleared as we write the results into some new external register provided for that purpose, originally containing 0's. This we do from t_M to t'_M by exchanging data in the empty external register with the M's output register.

We can now consider connecting such units in different ways. For example, the most obvious way is succession. If we want to do first M and then N we can connect the terminal side of one to the starting side of the other as in Fig. 11, to produce a new effective operator K, and the Hamiltonian then for H_K is

$$H_K(s_K, t_K) = H_M(s_K, t) + H_N(t, t_K)$$

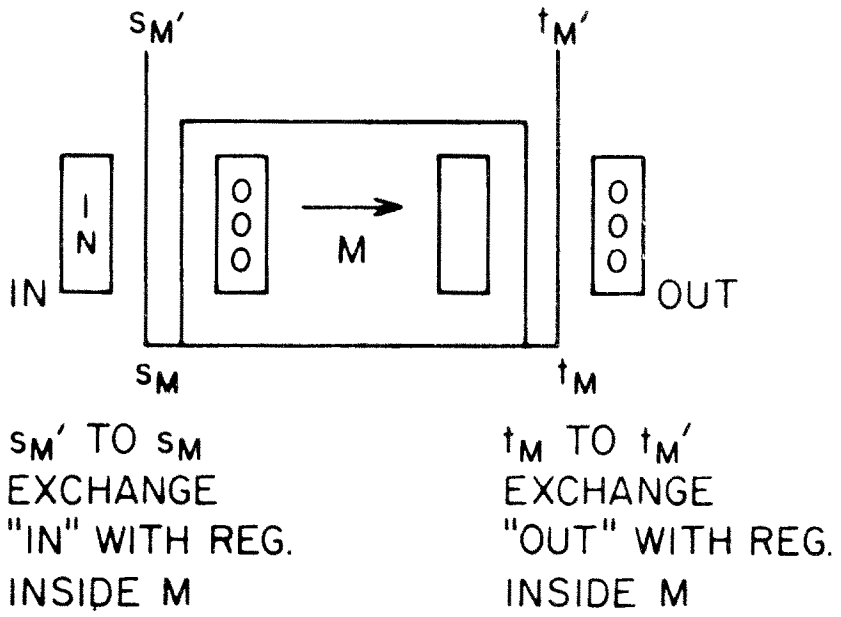


Fig. 10. Piece with external input and output.

The general conditional, if $a=1$ do M, but if $a=0$ do N, can be made, as in Fig. 12. For this

$$H_{\text{cond}}(s_c, t_c) = (s_M^* a s_c + t_c^* a^* t_M + s_N^* a^* s_c + t_c^* a t_N + \text{c.c.}) + H_M(s_M, t_M) + H_N(s_N, t_N)$$

The CONTROLLED NOT is the special case of this with $M = \text{NOT } b$ for which H is

$$H_{\text{NOT}b}(s, t) = s^*(b + b^*)\, t + \text{c.c.}$$

and N is no operation s^*t.

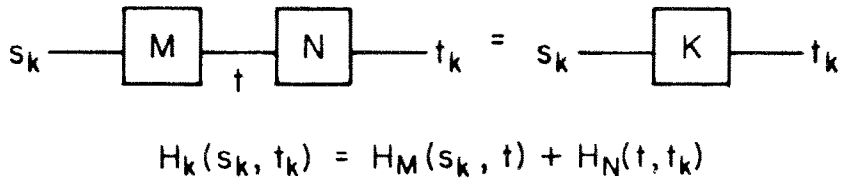


Fig. 11. Operations in succession.

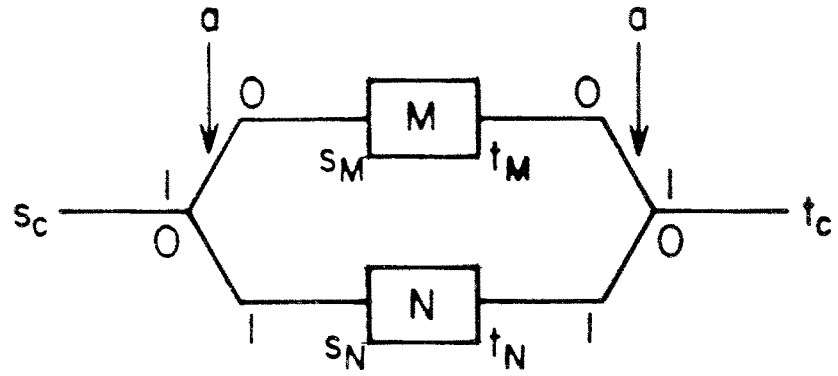


Fig. 12. Conditional if $a = 1$ then M, else N.

As another example, we can deal with a garbage clearer (previously described in Fig. 6) not by making two machines, a machine and its inverse, but by using the same machine and then sending the data back to the machine in the opposite direction, using our switch (see Fig. 13).

Suppose in this system we have a special flag which is originally always set to 0. We also suppose we have the input data in an external register, an empty external register available to hold the output, and the machine registers all empty (containing 0's). We come on the starting line s.

The first thing we do is to copy (using CONTROLLED NOT's) our external input into M. Then M operates, and the cursor goes on the top line in our drawing. It copies the output out of M into the external output register. M now contains garbage. Next it changes f to NOT f, comes down on the other line of the switch, backs out through M clearing the garbage, and uncopies the input again.

When you copy data and do it again, you reduce one of the registers to 0, the register into which you coied the first time. After the coying, it goes out (since f is now changed) on the other line where we restore f to 0

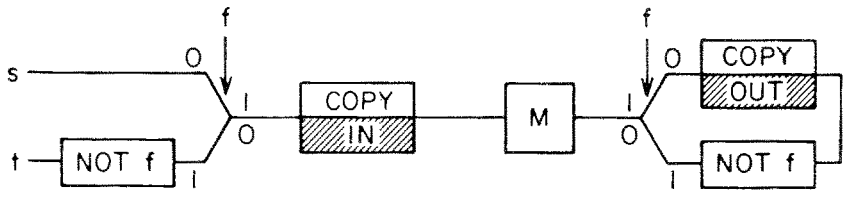


Fig. 13. Garbage clearer.

and come out at t. So between s and t we have a new piece of equipment, which has the following properties.

When its starts, we have, in a register called IN, the input data. In an external register which we call OUT, we have 0's. There is an internal flag set at 0, and the box, M, is empty of all data. At the termination of this, at t, the input register still contains the input data, and the output register contains the output of the effort of the operator M. M, however, is still empty, and the flag f is reset to 0.

Also important in computer programs is the ability to use the same subroutine several times. Of course, from a logical point of view, that can be done by writing that bit of program over and over again, each time it is to be used, but in a practical computer, it is much better if we could build that section of the computer which does a particular operation, just once, and use that section again and again.

To show the possibilities, here, first just suppose we have an operation we simply wish to repeat twice in succession (see Fig. 14). We start at s with the flag a in the condition 0, and thus we come along the line, and the first thing that happens is we change the value of a. Next we do the operation M. Now, because we changed a, instead of coming out at the top line where we went in, we come out at the bottom line, which recirculates the program back into changing a again; it restores it.

This time as we go through M, we come out and we have the a to follow on the uper line, and thus come out at the terminal, t. The Hamiltonian for this is

$$H_{MM}(s, t) = (s_N^* a^* s + s_M^*(a^* + a)\, s_N + x^* a^* t_M + s_N^* a x + t^* a t_M + \text{c.c.}) + H_M(s_M, t_M)$$

Using this switching circuit a number of times, of course, we can repeat an operation several times. For example, using the same idea three

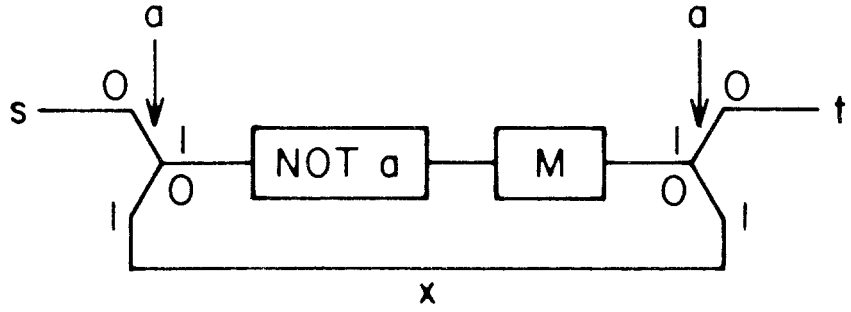


Fig. 14. Do M twice.

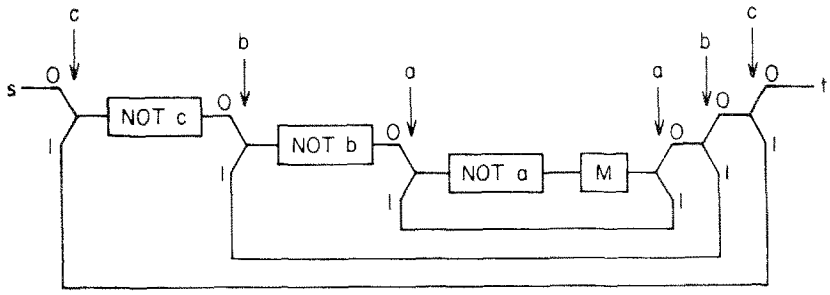


Fig. 15. Do M eight times.

times in succession, a nested succession, we can do an operation eight times, by the apparatus indicated in Fig. 15. In order to do so, we have three flags, a, b, and c. It is necessary to have flags when operations are done again for the reason that we must keep track of how many times its done and where we are in the program or we'll never be able to reverse things.

A subroutine in a normal computer can be used and emptied and used again without any record being kept of what happened. But here we have to keep a record and we do that with flags, of exactly where we are in the cycle of the use of the subroutine. If the subroutine is called from a certain place and has to go back to some other place, and another time is called, its origin and final destination are different, we have to know and keep track of where it came from and where it's supposed to go individually in each case, so more data have to be kept. Using a subroutine over and over in a reversible machine is only slightly harder than in a general machine. All these considerations appear in papers by Fredkin, Toffoli, and Bennett.

It is clear by the use of this switch, and successive uses of such switches in trees, that we would be able to steer data to any oint in a memory. A memory would simply be a place where there are registers into which you could copy data and then return the program. The cursor will

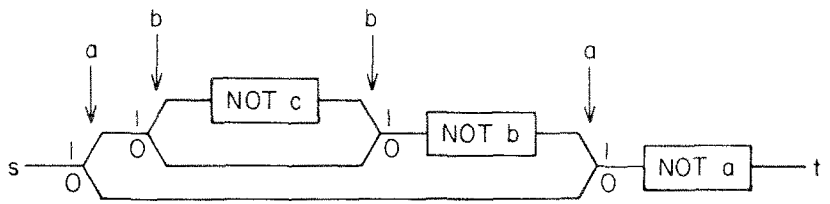


Fig. 16. Increment counter (3-bit).

have to follow the data along. I suppose there must be another set of tree switches set the oposite direction to carry the cursor out again, after copying the data so that the system remains reversible.

In Fig. 16 we show an incremental binary counter (of three bits a, b, c with c the most significant bit) which keeps track of how many net times the cursor has passed from s to t. These few examples should be enough to show that indeed we can construct all computer functions with our SWITCH and NOT. We need not follows this in more detail.

6. CONCLUSIONS

It's clear from these examples that this quentum machine has not really used many of the specific qualities of the differential equations of quantum mechanics.

What we have done is only to try to imitate as closely as possible the digital machine of conventional sequential architecture. It is analogous to the use of transistors in conventional machines, where we do not properly use all the analog continuum of the behavior of transistors, but just try to run them as saturated on or off digital devices so the logical analysis of the system behavior is easier. Furthermore, the system is absolutely sequential—for example, even in the comparison (exclusive or) of two k bit numbers, we must do each bit successively. What can be done, in these reversible quantum systems, to gain the speed available by concurrent operation has not been studied here.

Although, for theoretical and academic reasons, I have studied complete and reversible systems, if such tiny machines could become practical there is no reason why irreversible and entropy creating interactions cannot be made frequently during the course of operations of the machine.

For example, it might prove wise, in a long calculation, to ensure that the cursor has surely reached some oint and cannot be allowed to reverse again from there. Or, it may be found practical to connect irreversible memory storage (for items less frequently used) to reversible logic or short-term reversible storage registers, etc. Again, there is no reason we need to stick to chains of coupled sites for more distant communication where wires or light may be easier and faster.

At any rate, it seems that the laws of physics present no barrier to reducing the size of computers until bits are the size of atoms, and quantum behavior holds dominant sway.

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[3] I would like to thank T. Toffoli for his help with the references.

Space-Time Approach to Quantum Electrodynamics

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In this paper two things are done. (1) It is shown that a considerable simplification can be attained in writing down matrix elements for complex processes in electrodynamics. Further, a physical point of view is available which permits them to be written down directly for any specific problem. Being simply a restatement of conventional electrodynamics, however, the matrix elements diverge for complex processes. (2) Electrodynamics is modified by altering the interaction of electrons at short distances. All matrix elements are now finite, with the exception of those relating to problems of vacuum polarization. The latter are evaluated in a manner suggested by Pauli and Bethe, which gives finite results for these matrices also. The only effects sensitive to the modification are changes in mass and charge of the electrons. Such changes could not be directly observed. Phenomena directly observable, are insensitive to the details of the modification used (except at extreme energies). For such phenomena, a limit can be taken as the range of the modification goes to zero. The results then agree with those of Schwinger. A complete, unambiguous, and presumably consistent, method is therefore available for the calculation of all processes involving electrons and photons.

The simplification in writing the expressions results from an emphasis on the over-all space-time view resulting from a study of the solution of the equations of electrodynamics. The relation of this to the more conventional Hamiltonian point of view is discussed. It would be very difficult to make the modification which is proposed if one insisted on having the equations in Hamiltonian form.

The methods apply as well to charges obeying the Klein-Gordon equation, and to the various meson theories of nuclear forces. Illustrative examples are given. Although a modification like that used in electrodynamics can make all matrices finite for all of the meson theories, for some of the theories it is no longer true that all directly observable phenomena are insensitive to the details of the modification used.

The actual evaluation of integrals appearing in the matrix elements may be facilitated, in the simpler cases, by methods described in the appendix.

THIS paper should be considered as a direct continuation of a preceding one[1] (I) in which the motion of electrons, neglecting interaction, was analyzed, by dealing directly with the *solution* of the Hamiltonian differential equations. Here the same technique is applied to include interactions and in that way to express in simple terms the solution of problems in quantum electrodynamics.

For most practical calculations in quantum electrodynamics the solution is ordinarily expressed in terms of a matrix element. The matrix is worked out as an expansion in powers of $e^2/\hbar c$, the successive terms corresponding to the inclusion of an increasing number of virtual quanta. It appears that a considerable simplification can be achieved in writing down these matrix elements for complex processes. Furthermore, each term in the expansion can be written down and understood directly from a physical point of view, similar to the space-time view in I. It is the purpose of this paper to describe how this may be done. We shall also discuss methods of handling the divergent integrals which appear in these matrix elements.

The simplification in the formulae results mainly from the fact that previous methods unnecessarily separated into individual terms processes that were closely related physically. For example, in the exchange of a quantum between two electrons there were two terms depending on which electron emitted and which absorbed the quantum. Yet, in the virtual states considered, timing relations are not significant. Olny the order of operators in the matrix must be maintained. We have seen (I), that in addition, processes in which virtual pairs are produced can be combined with others in which only positive energy electrons are involved. Further, the effects of longitudinal and transverse waves can be combined together. The separations previously made were on an unrelativistic basis (reflected in the circumstance that apparently momentum but not energy is conserved in intermediate states). When the terms are combined and simplified, the relativistic invariance of the result is self-evident.

We begin by discussing the solution in space and time of the Schrödinger equation for particles interacting instantaneously. The results are immediately generalizable to delayed interactions of relativistic electrons and we represent in that way the laws of quantum electrodynamics. We can then see how the matrix element for any process can be written down directly. In particular, the self-energy expression is written down.

So far, nothing has been done other than a restatement of conventional electrodynamics in other terms. Therefore, the self-energy diverges. A modification[2] in interaction between charges is next made, and it is shown that the self-energy is made convergent and corresponds to a correction to the electron mass. After the mass correction is made, other real processes are finite and insensitive to the "width" of the cut-off in the interaction.[3]

Unfortunately, the modification proposed is not completely satisfactory theoretically (it leads to some difficulties of conservation of energy). It does, however, seem consistent and satisfactory to define the matrix

[1] R. P. Feynman, Phys. Rev. **76**, 749 (1949), hereafter called I.

[2] For a discussion of this modification in classical physics see R. P. Feynman, Phys. Rev. **74** 939 (1948), hereafter referred to as A.

[3] A brief summary of the methods and results will be found in R. P. Feynman, Phys. Rev. **74**, 1430 (1948), hereafter referred to as B.

element for all real processes as the limit of that computed here as the cut-off width goes to zero. A similar technique suggested by Pauli and by Bethe can be applied to problems of vacuum polarization (resulting in a renormalization of charge) but again a strict physical basis for the rules of convergence is not known.

After mass and charge renormalization, the limit of zero cut-off width can be taken for all real processes. The results are then equivalent to those of Schwinger[4] who does not make explicit use of the convergence factors. The method of Schwinger is to identify the terms corresponding to corrections in mass and charge and, previous to their evaluation, to remove them from the expressions for real processes. This has the advantage of showing that the results can be strictly independent of particular cut-off methods. On the other hand, many of the properties of the integrals are analyzed using formal properties of invariant propagation functions. But one of the properties is that the integrals are infinite and it is not clear to what extent this invalidates the demonstrations. A practical advantage of the present method is that ambiguities can be more easily resolved; simply by direct calculation of the otherwise divergent integrals. Nevertheless, it is not at all clear that the convergence factors do not upset the physical consistency of the theory. Although in the limit the two methods agree, neither method appears to be thoroughly satisfactory theoretically. Nevertheless, it does appear that we now have available a complete and definite method for the calculation of physical processes to any order in quantum electrodynamics.

Since we can write down the solution to any physical problem, we have a complete theory which could stand by itself. It will be theoretically incomplete, however, in two respects. First, although each term of increasing order in $e^2/\hbar c$ can be written down it would be desirable to see some way of expressing things in finite form to all orders in $e^2/\hbar c$ at once. Second, although it will be physically evident that the results obtained are equivalent to those obtained by conventional electrodynamics the mathematical proof of this is not included. Both of these limitations will be removed in a subsequent paper (see also Dyson[4]).

Briefly the genesis of this theory was this. The conventional electrodynamics was expressed in the Lagrangian form of quantum mechanics described in the Reviews of Modern Physics.[5] The motion of the field oscillators could be integrated out (as described in Section 13 of that paper), the result being an expression of the delayed interaction of the particles. Next the modification of the delta-function interaction could be made directly from the analogy to the classical case.[2] This was still not complete because the Lagrangian method had been worked out in detail only for particles obeying the non-relativistic Schrödinger equation. It was then modified in accordance with the requirements of the Dirac equation and the phenomenon of pair creation. This was made easier by the reinterpretation of the theory of holes (I). Finally for practical calculations the expressions were developed in a power series in $e^2/\hbar c$. It was apparent that each term in the series had a simple physical interpretation. Since the result was easier to understand than the derivation, it was thought best to publish the results first in this paper. Considerable time has been spent to make these first two papers as complete and as physically plausible as possible without relying on the Lagrangian method, because it is not generally familiar. It is realized that such a description cannot carry the conviction of truth which would accompany the derivation. On the other hand, in the interest of keeping simple things simple the derivation will appear in a separate paper.

The possible application of these methods to the various meson theories is discussed briefly. The formulas corresponding to a charge particle of zero spin moving in accordance with the Klein Gordon equation are also given. In an Appendix a method is given for calculating the integrals appearing in the matrix elements for the simpler processes.

The point of view which is taken here of the interaction of charges differs from the more usual point of view of field theory. Furthermore, the familiar Hamiltonian form of quantum mechanics must be compared to the over-all space-time view used here. The first section is, therefore, devoted to a discussion of the relations of these viewpoints.

1. COMPARISON WITH THE HAMILTONIAN METHOD

Electrodynamics can be looked upon in two equivalent and complementary ways. One is as the description of the behavior of a field (Maxwell's equations). The other is as a description of a direct interaction at a distance (albeit delayed in time) between charges (the solutions of Lienard and Wiechert). From the latter point of view light is considered as an interaction of the charges in the source with those in the absorber. This is an impractical point of view because many kinds of sources produce the same kind of effects. The field point of view separates these aspects into two simpler problems, production of light, and absorption of light. On the other hand, the field point of view is less practical when dealing with close collisions of particles (or their action on themselves). For here the source and absorber are not readily distinguishable, there is an intimate exchange of quanta. The fields are so closely determined by the motions of the particles that it is just as well not to separate the question into two problems but to consider the process as a direct interaction. Roughly, the field point of view is most practical for problems involv-

[4] J. Schwinger, Phys. Rev. **74**, 1439 (1948), Phys. Rev. **75**, 651 (1949). A proof of this equivalence is given by F. J. Dyson, Phys. Rev. **75**, 486 (1949).

[5] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948). The application to electrodynamics is described in detail by H. J. Groenewold, Koninklijke Nederlandsche Akademia van Weteschappen. Proceedings Vol. LII, 3 (226) 1949.

ing real quanta, while the interaction view is best for the discussion of the virtual quanta involved. We shall emphasize the interaction viewpoint in this paper, first because it is less familiar and therefore requires more discussion, and second because the important aspect in the problems with which we shall deal is the effect of virtual quanta.

The Hamiltonian method is not well adapted to represent the direct action at a distance between charges because that action is delayed. The Hamiltonian method represents the future as developing out of the present. If the values of a complete set of quantities are known now, their values can be computed at the next instant in time. If particles interact through a delayed interaction, however, one cannot predict the future by simply knowing the present motion of the particles. One would also have to know what the motions of the particles were in the past in view of the interaction this may have on the future motions. This is done in the Hamiltonian electrodynamics, of course, by requiring that one specify besides the present motion of the particles, the values of a host of new variables (the coordinates of the field oscillators) to keep track of that aspect of the past motions of the particles which determines their future behavior. The use of the Hamiltonian forces one to choose the field viewpoint rather than the interaction viewpoint.

In many problems, for example, the close collisions of particles, we are not interested in the precise temporal sequence of events. It is not of interest to be able to say how the situation would look at each instant of time during a collision and how it progresses from instant to instant. Such ideas are only useful for events taking a long time and for which we can readily obtain information during the intervening period. For collisions it is much easier to treat the process as a whole.[6] The Møller interaction matrix for the the collision of two electrons is not essentially more complicated than the nonrelativistic Rutherford formula, yet the mathematical machinery used to obtain the former from quantum electrodynamics is vastly more complicated than Schrödinger's equation with the e^2/r_{12} interaction needed to obtain the latter. The difference is only that in the latter the action is instantaneous so that the Hamiltonian method requires no extra variables, while in the former relativistic case it is delayed and the Hamiltonian method is very cumbersome.

We shall be discussing the solutions of equations rather than the time differential equations from which they come. We shall discover that the solutions, because of the over-all space-time view that they permit, are as easy to understand when interactions are delayed as when they are instantaneous.

As a further point, relativistic invariance will be self-evident. The Hamiltonian form of the equations develops the future from the instantaneous present. But for different observers in relative motion the instantaneous present is different, and corresponds to a different 3-dimensional cut of space-time. Thus the temporal analyses of different observers is different and their Hamiltonian equations are developing the process in different ways. These differences are irrelevant, however, for the solution is the same in any space time frame. By forsaking the Hamiltonian method, the wedding of relativity and quantum mechanics can be accomplished most naturally.

We illustrate these points in the next section by studying the solution of Schrödinger's equation for nonrelativistic particles interacting by an instantaneous Coulomb potential (Eq. 2). When the solution is modified to include the effects of delay in the interaction and the relativistic properties of the electrons we obtain an expression of the laws of quantum electrodynamics (Eq. 4).

2. THE INTERACTION BETWEEN CHARGES

We study by the same methods as in I, the interaction of two particles using the same notation as I. We start by considering the non-relativistic case described by the Schrödinger equation (I, Eq. 1). The wave function at a given time is a function $\psi(\mathbf{x}_a, \mathbf{x}_b, t)$ of the coordinates $\mathbf{x}_a$ and $\mathbf{x}_b$ of each particle. Thus call $K(\mathbf{x}_a, \mathbf{x}_b, t; \mathbf{x}_a', \mathbf{x}_b', t')$ the amplitude that particle a at $\mathbf{x}_a'$ at time t' will get to $\mathbf{x}_a$ at t while particle b at $\mathbf{x}_b'$ at t' gets to $\mathbf{x}_b$ at t. If the particles are free and do not interact this is

$$K(\mathbf{x}_a, \mathbf{x}_b, t; \mathbf{x}_a', \mathbf{x}_b', t') = K_{0a}(\mathbf{x}_a, t; \mathbf{x}_a', t') K_{0b}(\mathbf{x}_b, t; \mathbf{x}_b', t')$$

where K_{0a} is the K_0 function for particle a considered as free. In *this* case we can obviously define a quantity like K, but for which the time t need not be the same for particles a and b (likewise for t'); e.g.,

$$K_0(3, 4; 1, 2) = K_{0a}(3, 1) K_{0b}(4, 2) \tag{1}$$

can be thought of as the amplitude that particle a goes from $\mathbf{x}_1$ at t_1 to $\mathbf{x}_3$ at t_3 and that particle b goes from $\mathbf{x}_2$ at t_2 to $\mathbf{x}_4$ at t_4.

When the particles do interact, one can only define the quantity $K(3, 4; 1, 2)$ precisely if the interaction vanishes between t_1 and t_2 and also between t_3 and t_4. In a real physical system such is not the case. There is such an enormous advantage, however, to the concept that we shall continue to use it, imagining that we can neglect the effect of interactions between t_1 and t_2 and between t_3 and t_4. For practical problems this means choosing such long time intervals $t_3 - t_1$ and $t_4 - t_2$ that the extra interactions near the end points have small relative effects. As an example, in a scattering problem it may well be that the particles are so well separated initially and finally that the interaction at these times is negligible. Again energy values can be defined by the average rate of change of phase over such long time intervals that errors initially and finally can be neglected. Inasmuch as any physical problem can be defined in terms of scattering processes we do not lose much in

[6] This is the viewpoint of the theory of the S matrix of Heisenberg.

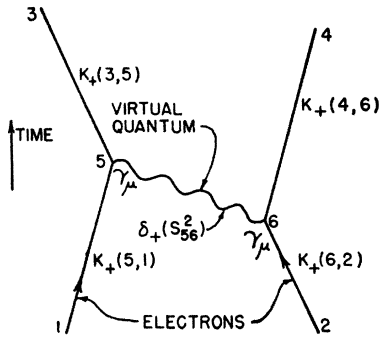

FIG. 1. The fundamental interaction Eq. (4). Exchange of one quantum between two electrons.

a general theoretical sense by this approximation. If it is not made it is not easy to study interacting particles relativistically, for there is nothing significant in choosing $t_1=t_3$ if $\mathbf{x}_1\neq\mathbf{x}_3$, as absolute simultaneity of events at a distance cannot be defined invariantly. It is essentially to avoid this approximation that the complicated structure of the older quantum electrodynamics has been built up. We wish to describe electrodynamics as a delayed interaction between particles. If we can make the approximation of assuming a meaning to $K(3,4;1,2)$ the results of this interaction can be expressed very simply.

To see how this may be done, imagine first that the interaction is simply that given by a Coulomb potential e^2/r where r is the distance between the particles. If this be turned on only for a very short time Δt_0 at time t_0, the first order correction to $K(3,4;1,2)$ can be worked out exactly as was Eq. (9) of I by an obvious generalization to two particles:

$$K^{(1)}(3,4;1,2)=-ie^2\int\int K_{0a}(3,5)K_{0b}(4,6)r_{56}^{-1}\times K_{0a}(5,1)K_{0b}(6,2)d^3\mathbf{x}_5d^3\mathbf{x}_6\Delta t_0,$$

where $t_5=t_6=t_0$. If now the potential were on at all times (so that strictly K is not defined unless $t_4=t_3$ and $t_1=t_2$), the first-order effect is obtained by integrating on t_0, which we can write as an integral over both t_5 and t_6 if we include a delta-function $\delta(t_5-t_6)$ to insure contribution only when $t_5=t_6$. Hence, the first-order effect of interaction is (calling $t_5-t_6=t_{56}$):

$$K^{(1)}(3,4;1,2)=-ie^2\int\int K_{0a}(3,5)K_{0b}(4,6)r_{56}^{-1}\times\delta(t_{56})K_{0a}(5,1)K_{0b}(6,2)d\tau_5d\tau_6, \quad (2)$$

where $d\tau=d^3\mathbf{x}dt$.

We know, however, in classical electrodynamics, that the Coulomb potential does not act instantaneously, but is delayed by a time r_{56}, taking the speed of light as unity. This suggests simply replacing $r_{56}^{-1}\delta(t_{56})$ in (2) by something like $r_{56}^{-1}\delta(t_{56}-r_{56})$ to represent the delay in the effect of b on a.

This turns out to be not quite right,[7] for when this interaction is represented by photons they must be of only positive energy, while the Fourier transform of $\delta(t_{56}-r_{56})$ contains frequencies of both signs. It should instead be replaced by $\delta_+(t_{56}-r_{56})$ where

$$\delta_+(x)=\int_0^\infty e^{-i\omega x}d\omega/\pi=\lim_{\epsilon\to0}\frac{(\pi i)^{-1}}{x-i\epsilon}=\delta(x)+(\pi ix)^{-1}. \quad (3)$$

This is to be averaged with $r_{56}^{-1}\delta_+(-t_{56}-r_{56})$ which arises when $t_5<t_6$ and corresponds to a emitting the quantum which b receives. Since

$$(2r)^{-1}(\delta_+(t-r)+\delta_+(-t-r))=\delta_+(t^2-r^2),$$

this means $r_{56}^{-1}\delta(t_{56})$ is replaced by $\delta_+(s_{56}^2)$ where $s_{56}^2=t_{56}^2-r_{56}^2$ is the square of the relativistically invariant interval between points 5 and 6. Since in classical electrodynamics there is also an interaction through the vector potential, the complete interaction (see A, Eq. (1)) should be $(1-(\mathbf{v}_5\cdot\mathbf{v}_6)\delta_+(s_{56}^2)$, or in the relativistic case,

$$(1-\boldsymbol{\alpha}_a\cdot\boldsymbol{\alpha}_b)\delta_+(s_{56}^2)=\beta_a\beta_b\gamma_{a\mu}\gamma_{b\mu}\delta_+(s_{56}^2).$$

Hence we have for electrons obeying the Dirac equation,

$$K^{(1)}(3,4;1,2)=-ie^2\int\int K_{+a}(3,5)K_{+b}(4,6)\gamma_{a\mu}\gamma_{b\mu}\times\delta_+(s_{56}^2)K_{+a}(5,1)K_{+b}(6,2)d\tau_5d\tau_6, \quad (4)$$

where $\gamma_{a\mu}$ and $\gamma_{b\mu}$ are the Dirac matrices applying to the spinor corresponding to particles a and b, respectively (the factor $\beta_a\beta_b$ being absorbed in the definition, I Eq. (17), of K_+).

This is our fundamental equation for electrodynamics. It describes the effect of exchange of one quantum (therefore first order in e^2) between two electrons. It will serve as a prototype enabling us to write down the corresponding quantities involving the exchange of two or more quanta between two electrons or the interaction of an electron with itself. It is a consequence of conventional electrodynamics. Relativistic invariance is clear. Since one sums over μ it contains the effects of both longitudinal and transverse waves in a relativistically symmetrical way.

We shall now interpret Eq. (4) in a manner which will permit us to write down the higher order terms. It can be understood (see Fig. 1) as saying that the amplitude for "a" to go from 1 to 3 and "b" to go from 2 to 4 is altered to first order because they can exchange a quantum. Thus, "a" can go to 5 (amplitude $K_+(5,1)$)

[7] It, and a like term for the effect of a on b, leads to a theory which, in the classical limit, exhibits interaction through half-advanced and half-retarded potentials. Classically, this is equivalent to purely retarded effects within a closed box from which no light escapes (e.g., see A, or J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. 17, 157 (1945)). Analogous theorems exist in quantum mechanics but it would lead us too far astray to discuss them now.

emit a quantum (longitudinal, transverse, or scalar $\gamma_{a\mu}$) and then proceed to 3 ($K_+(3, 5)$). Meantime "b" goes to 6 ($K_+(6, 2)$), absorbs the quantum ($\gamma_{b\mu}$) and proceeds to 4 ($K_+(4, 6)$). The quantum meanwhile proceeds from 5 to 6, which it does with amplitude $\delta_+(s_{56}^2)$. We must sum over all the possible quantum polarizations μ and positions and times of emission 5, and of absorption 6. Actually if $t_5 > t_6$ it would be better to say that "a" absorbs and "b" emits but no attention need be paid to these matters, as all such alternatives are automatically contained in (4).

The correct terms of higher order in e^2 or involving larger numbers of electrons (interacting with themselves or in pairs) can be written down by the same kind of reasoning. They will be illustrated by examples as we proceed. In a succeeding paper they will all be deduced from conventional quantum electrodynamics.

Calculation, from (4), of the transition element between positive energy free electron states gives the Möller scattering of two electrons, when account is taken of the Pauli principle.

The exclusion principle for interacting charges is handled in exactly the same way as for non-interacting charges (I). For example, for two charges it requires only that one calculate $K(3, 4; 1, 2) - K(4, 3; 1, 2)$ to get the net amplitude for arrival of charges at 3 and 4. It is disregarded in intermediate states. The interference effects for scattering of electrons by positrons discussed by Bhabha will be seen to result directly in this formulation. The formulas are interpreted to apply to positrons in the manner discussed in I.

As our primary concern will be for processes in which the quanta are virtual we shall not include here the detailed analysis of processes involving real quanta in initial or final state, and shall content ourselves by only stating the rules applying to them.[8] The result of the analysis is, as expected, that they can be included by the same line of reasoning as is used in discussing the virtual processes, provided the quantities are normalized in the usual manner to represent single quanta. For example, the amplitude that an electron in going from 1 to 2 absorbs a quantum whose vector potential, suitably normalized, is $c_\mu \exp(-ik\cdot x) = C_\mu(x)$ is just the expression (I, Eq. (13)) for scattering in a potential with $\boldsymbol{A}$ (3) replaced by $\boldsymbol{C}$ (3). Each quantum interacts only once (either in emission or in absorption), terms like (I, Eq. (14)) occur only when there is more than one quantum involved. The Bose statistics of the quanta can, in all cases, be disregarded in intermediate states. The only effect of the statistics is to change the weight of initial or final states. If there are among quanta, in the initial state, some n which are identical then the weight of the state is $(1/n!)$ of what it would be if these quanta were considered as different (similarly for the final state).

3. THE SELF-ENERGY PROBLEM

Having a term representing the mutual interaction of a pair of charges, we must include similar terms to represent the interaction of a charge with itself. For under some circumstances what appears to be two distinct electrons may, according to I, be viewed also as a single electron (namely in case one electron was created in a pair with a positron destined to annihilate the other electron). Thus to the interaction between such electrons must correspond the possibility of the action of an electron on itself.[9]

This interaction is the heart of the self energy problem. Consider to first order in e^2 the action of an electron on itself in an otherwise force free region. The amplitude $K(2, 1)$ for a single particle to get from 1 to 2 differs from $K_+(2, 1)$ to first order in e^2 by a term

$$K^{(1)}(2, 1) = -ie^2 \int\int K_+(2, 4)\gamma_\mu K_+(4, 3)\gamma_\mu \times K_+(3, 1) d\tau_3 d\tau_4 \delta_+(s_{43}^2). \quad (6)$$

It arises because the electron instead of going from 1 directly to 2, may go (Fig. 2) first to 3, ($K_+(3, 1)$), emit a quantum (γ_μ), proceed to 4, ($K_+(4, 3)$), absorb it (γ_μ), and finally arrive at 2 ($K_+(2, 4)$). The quantum must go from 3 to 4 ($\delta_+(s_{43}^2)$).

This is related to the self-energy of a free electron in the following manner. Suppose initially, time t_1, we have an electron in state $f(1)$ which we imagine to be a positive energy solution of Dirac's equation for a free particle. After a long time $t_2 - t_1$ the perturbation will alter

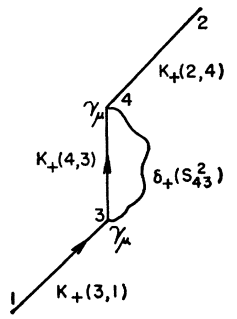

FIG. 2. Interaction of an electron with itself, Eq. (6).

[8] Although in the expressions stemming from (4) the quanta are virtual, this is not actually a theoretical limitation. One way to deduce the correct rules for real quanta from (4) is to note that in a closed system all quanta can be considered as virtual (i.e., they have a known source and are eventually absorbed) so that in such a system the present description is complete and equivalent to the conventional one. In particular, the relation of the Einstein A and B coefficients can be deduced. A more practical direct deduction of the expressions for real quanta will be given in the subsequent paper. It might be noted that (4) can be rewritten as describing the action on a, $K^{(1)}(3, 1) = i\int K_+(3, 5) \times \boldsymbol{A}(5) K_+(5, 1) d\tau_5$ of the potential $A_\mu(5) = e^2 \int K_+(4, 6)\delta_+(s_{56}^2)\gamma_\mu \times K_+(6, 2) d\tau_6$ arising from Maxwell's equations $-\Box^2 A_\mu = 4\pi j_\mu$ from a "current" $j_\mu(6) = e^2 K_+(4, 6)\gamma_\mu K_+(6, 2)$ produced by particle b in going from 2 to 4. This is virtue of the fact that δ_+ satisfies

$$-\Box_2^2 \delta_+(s_{21}^2) = 4\pi\delta(2, 1). \quad (5)$$

[9] These considerations make it appear unlikely that the contention of J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945), that electrons do not act on themselves, will be a successful concept in quantum electrodynamics.

the wave function, which can then be looked upon as a superposition of free particle solutions (actually it only contains f). The amplitude that $g(2)$ is contained is calculated as in (I, Eq. (21)). The diagonal element ($g=f$) is therefore

$$\int\int \bar{f}(2)\beta K^{(1)}(2,\,1)\beta f(1)d^3\mathbf{x}_1 d^3\mathbf{x}_2. \tag{7}$$

The time interval $T=t_2-t_1$ (and the spatial volume V over which one integrates) must be taken very large, for the expressions are only approximate (analogous to the situation for two interacting charges).[10] This is because, for example, we are dealing incorrectly with quanta emitted just before t_2 which would normally be reabsorbed at times after t_2.

If $K^{(1)}(2,\,1)$ from (6) is actually substituted into (7) the surface integrals can be performed as was done in obtaining I, Eq. (22) resulting in

$$-ie^2\int\int \bar{f}(4)\gamma_\mu K_+(4,\,3)\gamma_\mu f(3)\delta_+(s_{43}{}^2)d\tau_3 d\tau_4. \tag{8}$$

Putting for $f(1)$ the plane wave $u\exp(-ip\cdot x_1)$ where p_μ is the energy (p_4) and momentum of the electron ($p^2=m^2$), and u is a constant 4-index symbol, (8) becomes

$$-ie^2\int\int (\bar{u}\gamma_\mu K_+(4,\,3)\gamma_\mu u) \times\exp(ip\cdot(x_4-x_3))\delta_+(s_{43}{}^2)d\tau_3 d\tau_4,$$

the integrals extending over the volume V and time interval T. Since $K_+(4,\,3)$ depends only on the difference of the coordinates of 4 and 3, $x_{43\mu}$, the integral on 4 gives a result (except near the surfaces of the region) independent of 3. When integrated on 3, therefore, the result is of order VT. The effect is proportional to V, for the wave functions have been normalized to unit volume. If normalized to volume V, the result would simply be proportional to T. This is expected, for if the effect were equivalent to a change in energy ΔE, the amplitude for arrival in f at t_2 is altered by a factor $\exp(-i\Delta E(t_2-t_1))$, or to first order by the difference $-i(\Delta E)T$. Hence, we have

$$\Delta E=e^2\int (\bar{u}\gamma_\mu K_+(4,\,3)\gamma_\mu u)\exp(ip\cdot x_{43})\delta_+(s_{43}{}^2)d\tau_4, \tag{9}$$

integrated over all space-time $d\tau_4$. This expression will be simplified presently. In interpreting (9) we have tacitly assumed that the wave functions are normalized so that $(u^*u)=(\bar{u}\gamma_4 u)=1$. The equation may therefore be made independent of the normalization by writing the left side as $(\Delta E)(\bar{u}\gamma_4 u)$, or since $(\bar{u}\gamma_4 u)=(E/m)(\bar{u}u)$ and $m\Delta m=E\Delta E$, as $\Delta m(\bar{u}u)$ where Δm is an equivalent change in mass of the electron. In this form invariance is obvious.

One can likewise obtain an expression for the energy shift for an electron in a hydrogen atom. Simply replace K_+ in (8), by $K_+{}^{(V)}$, the exact kernel for an electron in the potential, $V=\beta e^2/r$, of the atom, and f by a wave function (of space and time) for an atomic state. In general the ΔE which results is not real. The imaginary part is negative and in $\exp(-i\Delta ET)$ produces an exponentially decreasing amplitude with time. This is because we are asking for the amplitude that an atom initially with no photon in the field, will still appear after time T with no photon. If the atom is in a state which can radiate, this amplitude must decay with time. The imaginary part of ΔE when calculated does indeed give the correct rate of radiation from atomic states. It is zero for the ground state and for a free electron.

In the non-relativistic region the expression for ΔE can be worked out as has been done by Bethe.[11] In the relativistic region (points 4 and 3 as close together as a Compton wave-length) the $K_+{}^{(V)}$ which should appear in (8) can be replaced to first order in V by K_+ plus $K_+{}^{(1)}(2,\,1)$ given in I, Eq. (13). The problem is then very similar to the radiationless scattering problem discussed below.

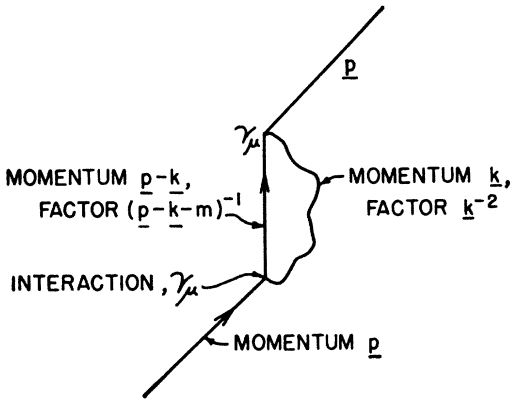

FIG. 3. Interaction of an electron with itself. Momentum space, Eq. (11).

4. EXPRESSION IN MOMENTUM AND ENERGY SPACE

The evaluation of (9), as well as all the other more complicated expressions arising in these problems, is very much simplified by working in the momentum and energy variables, rather than space and time. For this we shall need the Fourier Transform of $\delta_+(s_{21}{}^2)$ which is

$$-\delta_+(s_{21}{}^2)=\pi^{-1}\int \exp(-ik\cdot x_{21})k^{-2}d^4k, \tag{10}$$

which can be obtained from (3) and (5) or from I, Eq. (32) noting that $I_+(2,\,1)$ for $m^2=0$ is $\delta_+(s_{21}{}^2)$ from

[10] This is discussed in reference 5 in which it is pointed out that the concept of a wave function loses accuracy if there are delayed self-actions.

[11] H. A. Bethe, Phys. Rev. **72**, 339 (1947).

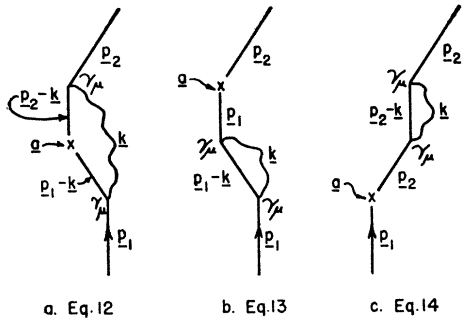

FIG. 4. Radiative correction to scattering, momentum space.

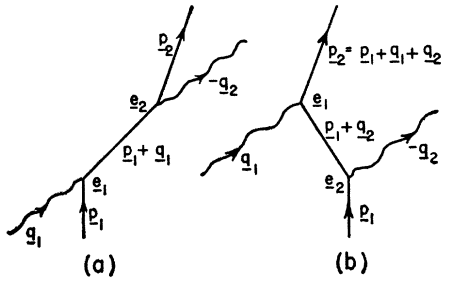

FIG. 5. Compton scattering, Eq. (15).

I, Eq. (34). The k^{-2} means $(k\cdot k)^{-1}$ or more precisely the limit as $\delta\to 0$ of $(k\cdot k+i\delta)^{-1}$. Further d^4k means $(2\pi)^{-2}dk_1dk_2dk_3dk_4$. If we imagine that quanta are particles of zero mass, then we can make the general rule that all poles are to be resolved by considering the masses of the particles and quanta to have infinitesimal negative imaginary parts.

Using these results we see that the self-energy (9) is the matrix element between $\bar{u}$ and u of the matrix

$$(e^2/\pi i)\int \gamma_\mu(\boldsymbol{p}-\boldsymbol{k}-m)^{-1}\gamma_\mu k^{-2}d^4k, \qquad (11)$$

where we have used the expression (I, Eq. (31)) for the Fourier transform of K_+. This form for the self-energy is easier to work with than is (9).

The equation can be understood by imagining (Fig. 3) that the electron of momentum $\boldsymbol{p}$ emits (γ_μ) a quantum of momentum $\boldsymbol{k}$, and makes its way now with momentum $\boldsymbol{p}-\boldsymbol{k}$ to the next event (factor $(\boldsymbol{p}-\boldsymbol{k}-m)^{-1}$) which is to absorb the quantum (another γ_μ). The amplitude of propagation of quanta is k^{-2}. (There is a factor $e^2/\pi i$ for each virtual quantum). One integrates over all quanta. The reason an electron of momentum $\boldsymbol{p}$ propagates as $1/(\boldsymbol{p}-m)$ is that this operator is the reciprocal of the Dirac equation operator, and we are simply solving this equation. Likewise light goes as $1/k^2$, for this is the reciprocal D'Alembertian operator of the wave equation of light. The first γ_μ represents the current which generates the vector potential, while the second is the velocity operator by which this potential is multiplied in the Dirac equation when an external field acts on an electron.

Using the same line of reasoning, other problems may be set up directly in momentum space. For example, consider the scattering in a potential $A=A_\mu\gamma_\mu$ varying in space and time as $\boldsymbol{a}\exp(-iq\cdot x)$. An electron initially in state of momentum $\boldsymbol{p}_1=p_{1\mu}\gamma_\mu$ will be deflected to state $\boldsymbol{p}_2$ where $\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$. The zero-order answer is simply the matrix element of $\boldsymbol{a}$ between states 1 and 2. We next ask for the first order (in e^2) radiative correction due to virtual radiation of one quantum. There are several ways this can happen. First for the case illustrated in Fig. 4(a), find the matrix:

$$(e^2/\pi i)\int \gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}-m)^{-1}\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}-m)^{-1}\gamma_\mu k^{-2}d^4k. \qquad (12)$$

For in this case, first[12] a quantum of momentum $\boldsymbol{k}$ is emitted (γ_μ), the electron then having momentum $\boldsymbol{p}_1-\boldsymbol{k}$ and hence propagating with factor $(\boldsymbol{p}_1-\boldsymbol{k}-m)^{-1}$. Next it is scattered by the potential (matrix $\boldsymbol{a}$) receiving additional momentum $\boldsymbol{q}$, propagating on then (factor $(\boldsymbol{p}_2-\boldsymbol{k}-m)^{-1}$) with the new momentum until the quantum is reabsorbed (γ_μ). The quantum propagates from emission to absorption (k^{-2}) and we integrate over all quanta (d^4k), and sum on polarization μ. When this is integrated on k_4, the result can be shown to be exactly equal to the expressions (16) and (17) given in B for the same process, the various terms coming from residues of the poles of the integrand (12).

Or again if the quantum is both emitted and reabsorbed before the scattering takes place one finds (Fig. 4(b))

$$(e^2/\pi i)\int \boldsymbol{a}(\boldsymbol{p}_1-m)^{-1}\gamma_\mu(\boldsymbol{p}_1-\boldsymbol{k}-m)^{-1}\gamma_\mu k^{-2}d^4k, \qquad (13)$$

or if both emission and absorption occur after the scattering, (Fig. 4(c))

$$(e^2/\pi i)\int \gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}-m)^{-1}\gamma_\mu(\boldsymbol{p}_2-m)^{-1}\boldsymbol{a}k^{-2}d^4k. \qquad (14)$$

These terms are discussed in detail below.

We have now achieved our simplification of the form of writing matrix elements arising from virtual processes. Processes in which a number of real quanta is given initially and finally offer no problem (assuming correct normalization). For example, consider the Compton effect (Fig. 5(a)) in which an electron in state $\boldsymbol{p}_1$ absorbs a quantum of momentum $\boldsymbol{q}_1$, polarization vector $e_{1\mu}$ so that its interaction is $e_{1\mu}\gamma_\mu=\boldsymbol{e}_1$, and emits a second quantum of momentum $-\boldsymbol{q}_2$, polarization $\boldsymbol{e}_2$ to arrive in final state of momentum $\boldsymbol{p}_2$. The matrix for

[12] First, next, etc., here refer not to the order in true time but to the succession of events along the trajectory of the electron. That is, more precisely, to the order of appearance of the matrices in the expressions.

this process is $e_2(p_1+q_1-m)^{-1}e_1$. The total matrix for the Compton effect is, then,

$$e_2(p_1+q_1-m)^{-1}e_1+e_1(p_1+q_2-m)^{-1}e_2, \quad (15)$$

the second term arising because the emission of e_2 may also precede the absorption of e_1 (Fig. 5(b)). One takes matrix elements of this between initial and final electron states ($p_1+q_1=p_2-q_2$), to obtain the Klein Nishina formula. Pair annihilation with emission of two quanta, etc., are given by the same matrix, positron states being those with negative time component of p. Whether quanta are absorbed or emitted depends on whether the time component of q is positive or negative.

5. THE CONVERGENCE OF PROCESSES WITH VIRTUAL QUANTA

These expressions are, as has been indicated, no more than a re-expression of conventional quantum electrodynamics. As a consequence, many of them are meaningless. For example, the self-energy expression (9) or (11) gives an infinite result when evaluated. The infinity arises, apparently, from the coincidence of the δ-function singularities in $K_+(4, 3)$ and $\delta_+(s_{43}^2)$. Only at this point is it necessary to make a real departure from conventional electrodynamics, a departure other than simply rewriting expressions in a simpler form.

We desire to make a modification of quantum electrodynamics analogous to the modification of classical electrodynamics described in a previous article, A. There the $\delta(s_{12}^2)$ appearing in the action of interaction was replaced by $f(s_{12}^2)$ where $f(x)$ is a function of small width and great height.

The obvious corresponding modification in the quantum theory is to replace the $\delta_+(s^2)$ appearing the quantum mechanical interaction by a new function $f_+(s^2)$. We can postulate that if the Fourier transform of the classical $f(s_{12}^2)$ is the integral over all k of $F(k^2)\exp(-ik\cdot x_{12})d^4k$, then the Fourier transform of $f_+(s^2)$ is the same integral taken over only positive frequencies k_4 for $t_2>t_1$ and over only negative ones for $t_2<t_1$ in analogy to the relation of $\delta_+(s^2)$ to $\delta(s^2)$. The function $f(s^2)=f(x\cdot x)$ can be written* as

$$f(x\cdot x)=(2\pi)^{-2}\int_{k_4=0}^{\infty}\int \sin(k_4|x_4|)\times\cos(\mathbf{K}\cdot\mathbf{x})dk_4d^3\mathbf{K}g(k\cdot k),$$

where $g(k\cdot k)$ is k_4^{-1} times the density of oscillators and may be expressed for positive k_4 as (A, Eq. (16))

$$g(k^2)=\int_0^{\infty}(\delta(k^2)-\delta(k^2-\lambda^2))G(\lambda)d\lambda,$$

where $\int_0^{\infty}G(\lambda)d\lambda=1$ and G involves values of λ large compared to m. This simply means that the amplitude for propagation of quanta of momentum k is

$$-F_+(k^2)=\pi^{-1}\int_0^{\infty}(k^{-2}-(k^2-\lambda^2)^{-1})G(\lambda)d\lambda,$$

rather than k^{-2}. That is, writing $F_+(k^2)=-\pi^{-1}k^{-2}C(k^2)$,

$$-f_+(s_{12}^2)=\pi^{-1}\int \exp(-ik\cdot x_{12})k^{-2}C(k^2)d^4k. \quad (16)$$

Every integral over an intermediate quantum which previously involved a factor d^4k/k^2 is now supplied with a convergence factor $C(k^2)$ where

$$C(k^2)=\int_0^{\infty}-\lambda^2(k^2-\lambda^2)^{-1}G(\lambda)d\lambda. \quad (17)$$

The poles are defined by replacing k^2 by $k^2+i\delta$ in the limit $\delta\to 0$. That is λ^2 may be assumed to have an infinitesimal negative imaginary part.

The function $f_+(s_{12}^2)$ may still have a discontinuity in value on the light cone. This is of no influence for the Dirac electron. For a particle satisfying the Klein Gordon equation, however, the interaction involves gradients of the potential which reinstates the δ function if f has discontinuities. The condition that f is to have no discontinuity in value on the light cone implies $k^2C(k^2)$ approaches zero as k^2 approaches infinity. In terms of $G(\lambda)$ the condition is

$$\int_0^{\infty}\lambda^2G(\lambda)d\lambda=0. \quad (18)$$

This condition will also be used in discussing the convergence of vacuum polarization integrals.

The expression for the self-energy matrix is now

$$(e^2/\pi i)\int \gamma_\mu(p-k-m)^{-1}\gamma_\mu k^{-2}d^4kC(k^2), \quad (19)$$

which, since $C(k^2)$ falls off at least as rapidly as $1/k^2$, converges. For practical purposes we shall suppose hereafter that $C(k^2)$ is simply $-\lambda^2/(k^2-\lambda^2)$ implying that some average (with weight $G(\lambda)d\lambda$) over values of λ may be taken afterwards. Since in all processes the quantum momentum will be contained in at least one extra factor of the form $(p-k-m)^{-1}$ representing propagation of an electron while that quantum is in the field, we can expect all such integrals with their convergence factors to converge and that the result of all such processes will now be finite and definite (excepting the processes with closed loops, discussed below, in which the diverging integrals are over the momenta of the electrons rather than the quanta).

The integral of (19) with $C(k^2)=-\lambda^2(k^2-\lambda^2)^{-1}$ noting that $p^2=m^2$, $\lambda\gg m$ and dropping terms of order m/λ, is (see Appendix A)

$$(e^2/2\pi)[4m(\ln(\lambda/m)+\tfrac{1}{2})-p(\ln(\lambda/m)+5/4)]. \quad (20)$$

* This relation is given incorrectly in A, equation just preceding 16.

When applied to a state of an electron of momentum $\boldsymbol{p}$ satisfying $\boldsymbol{p}u=mu$, it gives for the change in mass (as in B, Eq. (9))

$$\Delta m=m(e^2/2\pi)(3\ln(\lambda/m)+\tfrac{3}{4}). \tag{21}$$

6. RADIATIVE CORRECTIONS TO SCATTERING

We can now complete the discussion of the radiative corrections to scattering. In the integrals we include the convergence factor $C(\boldsymbol{k}^2)$, so that they converge for large $\boldsymbol{k}$. Integral (12) is also not convergent because of the well-known infra-red catastrophy. For this reason we calculate (as discussed in B) the value of the integral assuming the photons to have a small mass $\lambda_{\min}\ll m\ll\lambda$. The integral (12) becomes

$$(e^2/\pi i)\int\gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}-m)^{-1}\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}-m)^{-1}\times\gamma_\mu(\boldsymbol{k}^2-\lambda_{\min}{}^2)^{-1}d^4kC(\boldsymbol{k}^2-\lambda_{\min}{}^2),$$

which when integrated (see Appendix B) gives $(e^2/2\pi)$ times

$$\left[2\left(\ln\frac{m}{\lambda_{\min}}-1\right)\left(1-\frac{2\theta}{\tan2\theta}\right)+\theta\tan\theta+\frac{4}{\tan2\theta}\int_0^\theta\alpha\tan\alpha d\alpha\right]\boldsymbol{a}+\frac{1}{4m}(\boldsymbol{qa}-\boldsymbol{aq})\frac{2\theta}{\sin2\theta}+r\boldsymbol{a}, \tag{22}$$

where $(\boldsymbol{q}^2)^{\frac{1}{2}}=2m\sin\theta$ and we have assumed the matrix to operate between states of momentum $\boldsymbol{p}_1$ and $\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$ and have neglected terms of order $\lambda_{\min}/m$, m/λ, and $\boldsymbol{q}^2/\lambda^2$. Here the only dependence on the convergence factor is in the term $r\boldsymbol{a}$, where

$$r=\ln(\lambda/m)+9/4-2\ln(m/\lambda_{\min}). \tag{23}$$

As we shall see in a moment, the other terms (13), (14) give contributions which just cancel the $r\boldsymbol{a}$ term. The remaining terms give for small $\boldsymbol{q}$,

$$(e^2/4\pi)\left(\frac{1}{2m}(\boldsymbol{qa}-\boldsymbol{aq})+\frac{4\boldsymbol{q}^2}{3m^2}\boldsymbol{a}\left(\ln\frac{m}{\lambda_{\min}}-\frac{3}{8}\right)\right), \tag{24}$$

which shows the change in magnetic moment and the Lamb shift as interpreted in more detail in B.[13]

We must now study the remaining terms (13) and (14). The integral on $\boldsymbol{k}$ in (13) can be performed (after multiplication by $C(\boldsymbol{k}^2)$) since it involves nothing but the integral (19) for the self-energy and the result is allowed to operate on the initial state u_1, (so that $\boldsymbol{p}_1u_1=mu_1$). Hence the factor following $\boldsymbol{a}(\boldsymbol{p}_1-m)^{-1}$ will be just Δm. But, if one now tries to expand $1/(\boldsymbol{p}_1-m)=(\boldsymbol{p}_1+m)/(\boldsymbol{p}_1{}^2-m^2)$ one obtains an infinite result, since $\boldsymbol{p}_1{}^2=m^2$. This is, however, just what is expected physically. For the quantum can be emitted and absorbed at any time previous to the scattering. Such a process has the effect of a change in mass of the electron in the state 1. It therefore changes the energy by ΔE and the amplitude to first order in ΔE by $-i\Delta E\cdot t$ where t is the time it is acting, which is infinite. That is, the major effect of this term would be canceled by the effect of change of mass Δm.

The situation can be analyzed in the following manner. We suppose that the electron approaching the scattering potential $\boldsymbol{a}$ has not been free for an infinite time, but at some time far past suffered a scattering by a potential $\boldsymbol{b}$. If we limit our discussion to the effects of Δm and of the virtual radiation of one quantum between two such scatterings each of the effects will be finite, though large, and their difference is determinate. The propagation from $\boldsymbol{b}$ to $\boldsymbol{a}$ is represented by a matrix

$$\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}, \tag{25}$$

in which one is to integrate possibly over $\boldsymbol{p}'$ (depending on details of the situation). (If the time is long between $\boldsymbol{b}$ and $\boldsymbol{a}$, the energy is very nearly determined so that $\boldsymbol{p}'{}^2$ is very nearly m^2.)

We shall compare the effect on the matrix (25) of the virtual quanta and of the change of mass Δm. The effect of a virtual quantum is

$$(e^2/\pi i)\int\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\gamma_\mu(\boldsymbol{p}'-\boldsymbol{k}-m)^{-1}\times\gamma_\mu(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}\boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2), \tag{26}$$

while that of a change of mass can be written

$$\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\Delta m(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}, \tag{27}$$

and we are interested in the difference (26)–(27). A simple and direct method of making this comparison is just to evaluate the integral on $\boldsymbol{k}$ in (26) and subtract from the result the expression (27) where Δm is given in (21). The remainder can be expressed as a multiple $-r(\boldsymbol{p}'{}^2)$ of the unperturbed amplitude (25);

$$-r(\boldsymbol{p}'{}^2)\boldsymbol{a}(\boldsymbol{p}'-m)^{-1}\boldsymbol{b}. \tag{28}$$

This has the same result (to this order) as replacing the potentials $\boldsymbol{a}$ and $\boldsymbol{b}$ in (25) by $(1-\frac{1}{2}r(\boldsymbol{p}'{}^2))\boldsymbol{a}$ and

[13] That the result given in B in Eq. (19) was in error was repeatedly pointed out to the author, in private communication, by V. F. Weisskopf and J. B. French, as their calculation, completed simultaneously with the author's early in 1948, gave a different result. French has finally shown that although the expression for the radiationless scattering B, Eq. (18) or (24) above is correct, it was incorrectly joined onto Bethe's non-relativistic result. He shows that the relation $\ln2k_{\max}-1=\ln\lambda_{\min}$ used by the author should have been $\ln2k_{\max}-5/6=\ln\lambda_{\min}$. This results in adding a term $-(1/6)$ to the logarithm in B, Eq. (19) so that the result now agrees with that of J. B. French and V. F. Weisskopf, Phys. Rev. **75**, 1240 (1949) and N. H. Kroll and W. E. Lamb, Phys. Rev. **75**, 388 (1949). The author feels unhappily responsible for the very considerable delay in the publication of French's result occasioned by this error. This footnote is appropriately numbered.

$(1-\frac{1}{2}r(\not{p}'^2))b$. In the limit, then, as $\not{p}'^2\to m^2$ the net effect on the scattering is $-\frac{1}{2}ra$ where r, the limit of $r(\not{p}'^2)$ as $\not{p}'^2\to m^2$ (assuming the integrals have an infrared cut-off), turns out to be just equal to that given in (23). An equal term $-\frac{1}{2}ra$ arises from virtual transitions after the scattering (14) so that the entire ra term in (22) is canceled.

The reason that r is just the value of (12) when $q^2=0$ can also be seen without a direct calculation as follows: Let us call $\not{p}$ the vector of length m in the direction of $\not{p}'$ so that if $\not{p}'^2=m(1+\epsilon)^2$ we have $\not{p}'=(1+\epsilon)\not{p}$ and we take ϵ as very small, being of order T^{-1} where T is the time between the scatterings b and a. Since $(\not{p}'-m)^{-1}=(\not{p}'+m)/(\not{p}'^2-m^2)\approx(\not{p}+m)/2m^2\epsilon$, the quantity (25) is of order ϵ^{-1} or T. We shall compute corrections to it only to its own order (ϵ^{-1}) in the limit $\epsilon\to 0$. The term (27) can be written approximately[14] as

$$(e^2/\pi i)\int a(\not{p}'-m)^{-1}\gamma_\mu(\not{p}-\not{k}-m)^{-1}\times\gamma_\mu(\not{p}'-m)^{-1}b\not{k}^{-2}d^4kC(k^2),$$

using the expression (19) for Δm. The net of the two effects is therefore approximately[15]

$$-(e^2/\pi i)\int a(\not{p}'-m)^{-1}\gamma_\mu(\not{p}-\not{k}-m)^{-1}\epsilon\not{p}(\not{p}-\not{k}-m)^{-1}\times\gamma_\mu(\not{p}'-m)^{-1}b\not{k}^{-2}d^4kC(k^2),$$

a term now of order $1/\epsilon$ (since $(\not{p}'-m)^{-1}\approx(\not{p}+m)\times(2m^2\epsilon)^{-1}$) and therefore the one desired in the limit. Comparison to (28) gives for r the expression

$$(\not{p}_1+m/2m)\int\gamma_\mu(\not{p}_1-\not{k}-m)^{-1}(\not{p}_1m^{-1})(\not{p}_1-\not{k}-m)^{-1}\times\gamma_\mu\not{k}^{-2}d^4kC(k^2). \quad (29)$$

The integral can be immediately evaluated, since it is the same as the integral (12), but with $q=0$, for a replaced by $\not{p}_1/m$. The result is therefore $r\cdot(\not{p}_1/m)$ which when acting on the state u_1 is just r, as $\not{p}_1u_1=mu_1$. For the same reason the term $(\not{p}_1+m)/2m$ in (29) is effectively 1 and we are left with $-r$ of (23).[16]

In more complex problems starting with a free electron the same type of term arises from the effects of a virtual emission and absorption both previous to the other processes. They, therefore, simply lead to the same factor r so that the expression (23) may be used directly and these renormalization integrals need not be computed afresh for each problem.

In this problem of the radiative corrections to scattering the net result is insensitive to the cut-off. This means, of course, that by a simple rearrangement of terms previous to the integration we could have avoided the use of the convergence factors completely (see for example Lewis[17]). The problem was solved in the manner here in order to illustrate how the use of such convergence factors, even when they are actually unnecessary, may facilitate analysis somewhat by removing the effort and ambiguities that may be involved in trying to rearrange the otherwise divergent terms.

The replacement of δ_+ by f_+ given in (16), (17) is not determined by the analogy with the classical problem. In the classical limit only the real part of δ_+ (i.e., just δ) is easy to interpret. But by what should the imaginary part, $1/(\pi is^2)$, of δ_+ be replaced? The choice we have made here (in defining, as we have, the location of the poles of (17)) is arbitrary and almost certainly incorrect. If the radiation resistance is calculated for an atom, as the imaginary part of (8), the result depends slightly on the function f_+. On the other hand the light radiated at very large distances from a source is independent of f_+. The total energy absorbed by distant absorbers will not check with the energy loss of the source. We are in a situation analogous to that in the classical theory if the entire f function is made to contain only retarded contributions (see A, Appendix). One desires instead the analogue of $\langle F\rangle_{\text{ret}}$ of A. This problem is being studied.

One can say therefore, that this attempt to find a consistent modification of quantum electrodynamics is incomplete (see also the question of closed loops, below). For it could turn out that any correct form of f_+ which will guarantee energy conservation may at the same time not be able to make the self-energy integral finite. The desire to make the methods of simplifying the calculation of quantum electrodynamic processes more widely available has prompted this publication before an analysis of the correct form for f_+ is complete. One might try to take the position that, since the energy discrepancies discussed vanish in the limit $\lambda\to\infty$, the correct physics might be considered to be that obtained by letting $\lambda\to\infty$ after mass renormalization. I have no proof of the mathematical consistency of this procedure, but the presumption is very strong that it is satisfactory. (It is also strong that a satisfactory form for f_+ can be found.)

7. THE PROBLEM OF VACUUM POLARIZATION

In the analysis of the radiative corrections to scattering one type of term was not considered. The potential

[14] The expression is not exact because the substitution of Δm by the integral in (19) is valid only if $\not{p}$ operates on a state such that $\not{p}$ can be replaced by m. The error, however, is of order $a(\not{p}'-m)^{-1}(\not{p}-m)(\not{p}'-m)^{-1}b$ which is $a((1+\epsilon)\not{p}+m)(\not{p}-m)\times((1+\epsilon)\not{p}+m)\not{p}(2\epsilon+\epsilon^2)^{-2}m^{-4}$. But since $\not{p}^2=m^2$, we have $\not{p}(\not{p}-m)=-m(\not{p}-m)=(\not{p}-m)\not{p}$ so the net result is approximately $a(\not{p}-m)b/4m^2$ and is not of order $1/\epsilon$ but smaller, so that its effect drops out in the limit.

[15] We have used, to first order, the general expansion (valid for any operators A, B)

$$(A+B)^{-1}=A^{-1}-A^{-1}BA^{-1}+A^{-1}BA^{-1}BA^{-1}-\cdots$$

with $A=\not{p}-\not{k}-m$ and $B=\not{p}'-\not{p}=\epsilon\not{p}$ to expand the difference of $(\not{p}'-\not{k}-m)^{-1}$ and $(\not{p}-\not{k}-m)^{-1}$.

[16] The renormalization terms appearing B, Eqs. (14), (15) when translated directly into the present notation do not give twice (29) but give this expression with the central $\not{p}_1m^{-1}$ factor replaced by $m\gamma_4/E_1$ where $E_1=p_{1\mu}$ for $\mu=4$. When integrated it therefore gives $ra((\not{p}_1+m)/2m)(m\gamma_4/E_1)$ or $ra-ra(m\gamma_4/E_1)(\not{p}_1-m)/2m$. (Since $\not{p}_1\gamma_4+\gamma_4\not{p}_1=2E_1$) which gives just ra, since $\not{p}_1u_1=mu_1$.

[17] H. W. Lewis, Phys. Rev. 73, 173 (1948).

which we can assume to vary as $a_\mu \exp(-iq\cdot x)$ creates a pair of electrons (see Fig. 6), momenta $\boldsymbol{p}_a$, $-\boldsymbol{p}_b$. This pair then reannihilates, emitting a quantum $\boldsymbol{q}=\boldsymbol{p}_b-\boldsymbol{p}_a$, which quantum scatters the original electron from state 1 to state 2. The matrix element for this process (and the others which can be obtained by rearranging the order in time of the various events) is

$$-(e^2/\pi i)(\bar{u}_2\gamma_\mu u_1)\int Sp[(\boldsymbol{p}_a+\boldsymbol{q}-m)^{-1}\times\gamma_\nu(\boldsymbol{p}_a-m)^{-1}\gamma_\mu]d^4p_a\boldsymbol{q}^{-2}C(\boldsymbol{q}^2)a_\nu. \quad (30)$$

This is because the potential produces the pair with amplitude proportional to $a_\nu\gamma_\nu$, the electrons of momenta $\boldsymbol{p}_a$ and $-(\boldsymbol{p}_a+\boldsymbol{q})$ proceed from there to annihilate, producing a quantum (factor γ_μ) which propagates (factor $\boldsymbol{q}^{-2}C(\boldsymbol{q}^2)$) over to the other electron, by which it is absorbed (matrix element of γ_μ between states 1 and 2 of the original electron $(\bar{u}_2\gamma_\mu u_1)$). All momenta $\boldsymbol{p}_a$ and spin states of the virtual electron are admitted, which means the spur and the integral on d^4p_a are calculated.

One can imagine that the closed loop path of the positron-electron produces a current

$$4\pi j_\mu = J_{\mu\nu}a_\nu, \quad (31)$$

which is the source of the quanta which act on the second electron. The quantity

$$J_{\mu\nu}=-(e^2/\pi i)\int Sp[(\boldsymbol{p}+\boldsymbol{q}-m)^{-1}\times\gamma_\nu(\boldsymbol{p}-m)^{-1}\gamma_\mu]d^4p, \quad (32)$$

is then characteristic for this problem of polarization of the vacuum.

One sees at once that $J_{\mu\nu}$ diverges badly. The modification of δ to f alters the amplitude with which the current j_μ will affect the scattered electron, but it can do nothing to prevent the divergence of the integral (32) and of its effects.

One way to avoid such difficulties is apparent. From one point of view we are considering all routes by which a given electron can get from one region of space-time to another, i.e., from the source of electrons to the apparatus which measures them. From this point of view the closed loop path leading to (32) is unnatural. It might be assumed that the only paths of meaning are those which start from the source and work their way in a continuous path (possibly containing many time reversals) to the detector. Closed loops would be excluded. We have already found that this may be done for electrons moving in a fixed potential.

Such a suggestion must meet several questions, however. The closed loops are a consequence of the usual hole theory in electrodynamics. Among other things, they are required to keep probability conserved. The probability that no pair is produced by a potential is not unity and its deviation from unity arises from the imaginary part of $J_{\mu\nu}$. Again, with closed loops excluded, a pair of electrons once created cannot annihilate one another again, the scattering of light by light would be zero, etc. Although we are not experimentally sure of these phenomena, this does seem to indicate that the closed loops are necessary. To be sure, it is always possible that these matters of probability conservation, etc., will work themselves out as simply in the case of interacting particles as for those in a fixed potential. Lacking such a demonstration the presumption is that the difficulties of vacuum polarization are not so easily circumvented.[18]

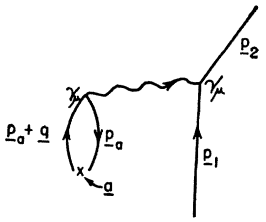

FIG. 6. Vacuum polarization effect on scattering, Eq. (30).

An alternative procedure discussed in B is to assume that the function $K_+(2,1)$ used above is incorrect and is to be replaced by a modified function K_+' having no singularity on the light cone. The effect of this is to provide a convergence factor $C(\boldsymbol{p}^2-m^2)$ for *every* integral over electron momenta.[19] This will multiply the integrand of (32) by $C(\boldsymbol{p}^2-m^2)C((\boldsymbol{p}+\boldsymbol{q})^2-m^2)$, since the integral was originally $\delta(\boldsymbol{p}_a-\boldsymbol{p}_b+\boldsymbol{q})d^4p_ad^4p_b$ and both $\boldsymbol{p}_a$ and $\boldsymbol{p}_b$ get convergence factors. The integral now converges but the result is unsatisfactory.[20]

One expects the current (31) to be conserved, that is $q_\mu j_\mu=0$ or $q_\mu J_{\mu\nu}=0$. Also one expects no current if a_ν is a gradient, or $a_\nu=q_\nu$ times a constant. This leads to the condition $J_{\mu\nu}q_\nu=0$ which is equivalent to $q_\mu J_{\mu\nu}=0$ since $J_{\mu\nu}$ is symmetrical. But when the expression (32) is integrated with such convergence factors it does not satisfy this condition. By altering the kernel from K to another, K', which does not satisfy the Dirac equation we have lost the gauge invariance, its consequent current conservation and the general consistency of the theory.

One can see this best by calculating $J_{\mu\nu}q_\nu$ directly from (32). The expression within the spur becomes $(\boldsymbol{p}+\boldsymbol{q}-m)^{-1}\boldsymbol{q}(\boldsymbol{p}-m)^{-1}\gamma_\mu$ which can be written as the difference of two terms: $(\boldsymbol{p}-m)^{-1}\gamma_\mu-(\boldsymbol{p}+\boldsymbol{q}-m)^{-1}\gamma_\mu$. Each of these terms would give the same result if the integration d^4p were without a convergence factor, for

[18] It would be very interesting to calculate the Lamb shift accurately enough to be sure that the 20 megacycles expected from vacuum polarization are actually present.

[19] This technique also makes self-energy and radiationless scattering integrals finite even without the modification of δ_+ to f_+ for the radiation (and the consequent convergence factor $C(\boldsymbol{k}^2)$ for the quanta). See B.

[20] Added to the terms given below (33) there is a term $\frac{1}{4}(\lambda^3-2\mu^2+\frac{1}{3}\boldsymbol{q}^2)\delta_{\mu\nu}$ for $C(\boldsymbol{k}^2)=-\lambda^2(\boldsymbol{k}^2-\lambda^2)^{-1}$, which is not gauge invariant. (In addition the charge renormalization has $-7/6$ added to the logarithm.)

the first can be converted into the second by a shift of the origin of p, namely $p'=p+q$. This does not result in cancelation in (32) however, for the convergence factor is altered by the substitution.

A method of making (32) convergent without spoiling the gauge invariance has been found by Bethe and by Pauli. The convergence factor for light can be looked upon as the result of superposition of the effects of quanta of various masses (some contributing negatively). Likewise if we take the factor $C(p^2-m^2)=-\lambda^2(p^2-m^2-\lambda^2)^{-1}$ so that $(p^2-m^2)^{-1}C(p^2-m^2)=(p^2-m^2)^{-1}-(p^2-m^2-\lambda^2)^{-1}$ we are taking the difference of the result for electrons of mass m and mass $(\lambda^2+m^2)^{\frac{1}{2}}$. But we have taken this difference for *each* propagation between interactions with photons. They suggest instead that once created with a certain mass the electron should continue to propagate with this mass through all the potential interactions until it closes its loop. That is if the quantity (32), integrated over some finite range of p, is called $J_{\mu\nu}(m^2)$ and the corresponding quantity over the same range of p, but with m replaced by $(m^2+\lambda^2)^{\frac{1}{2}}$ is $J_{\mu\nu}(m^2+\lambda^2)$ we should calculate

$$J_{\mu\nu}{}^P=\int_0^\infty [J_{\mu\nu}(m^2)-J_{\mu\nu}(m^2+\lambda^2)]G(\lambda)d\lambda, \quad (32')$$

the function $G(\lambda)$ satisfying $\int_0^\infty G(\lambda)d\lambda=1$ and $\int_0^\infty G(\lambda)\lambda^2 d\lambda=0$. Then in the expression for $J_{\mu\nu}{}^P$ the range of p integration can be extended to infinity as the integral now converges. The result of the integration using this method is the integral on $d\lambda$ over $G(\lambda)$ of (see Appendix C)

$$J_{\mu\nu}{}^P=-\frac{e^2}{\pi}(q_\mu q_\nu-\delta_{\mu\nu}q^2)\left(-\frac{1}{3}\ln\frac{\lambda^2}{m^2}-\left[\frac{4m^2+2q^2}{3q^2}\left(1-\frac{\theta}{\tan\theta}\right)-\frac{1}{9}\right]\right), \quad (33)$$

with $q^2=4m^2\sin^2\theta$.

The gauge invariance is clear, since $q_\mu(q_\mu q_\nu-q^2\delta_{\mu\nu})=0$. Operating (as it always will) on a potential of zero divergence the $(q_\mu q_\nu-\delta_{\mu\nu}q^2)a_\nu$ is simply $-q^2a_\mu$, the D'Alembertian of the potential, that is, the current producing the potential. The term $-\frac{1}{3}(\ln(\lambda^2/m^2))(q_\mu q_\nu-q^2\delta_{\mu\nu})$ therefore gives a current proportional to the current producing the potential. This would have the same effect as a change in charge, so that we would have a difference $\Delta(e^2)$ between e^2 and the experimentally observed charge, $e^2+\Delta(e^2)$, analogous to the difference between m and the observed mass. This charge depends logarithmically on the cut-off, $\Delta(e^2)/e^2=-(2e^2/3\pi)\ln(\lambda/m)$. After this renormalization of charge is made, no effects will be sensitive to the cut-off.

After this is done the final term remaining in (33), contains the usual effects[21] of polarization of the vacuum. It is zero for a free light quantum ($q^2=0$). For small q^2 it behaves as $(2/15)q^2$ (adding $-\frac{1}{5}$ to the logarithm in the Lamb effect). For $q^2>(2m)^2$ it is complex, the imaginary part representing the loss in amplitude required by the fact that the probability that no quanta are produced by a potential able to produce pairs ($(q^2)^{\frac{1}{2}}>2m$) decreases with time. (To make the necessary analytic continuation, imagine m to have a small negative imaginary part, so that $(1-q^2/4m^2)^{\frac{1}{2}}$ becomes $-i(q^2/4m^2-1)^{\frac{1}{2}}$ as q^2 goes from below to above $4m^2$. Then $\theta=\pi/2+iu$ where $\sinh u=+(q^2/4m^2-1)^{\frac{1}{2}}$, and $-1/\tan\theta=i\tanh u=+i(q^2-4m^2)^{\frac{1}{2}}(q^2)^{-\frac{1}{2}}$.)

Closed loops containing a number of quanta or potential interactions larger than two produce no trouble. Any loop with an odd number of interactions gives zero (I, reference 9). Four or more potential interactions give integrals which are convergent even without a convergence factor as is well known. The situation is analogous to that for self-energy. Once the simple problem of a single closed loop is solved there are no further divergence difficulties for more complex processes.[22]

8. LONGITUDINAL WAVES

In the usual form of quantum electrodynamics the longitudinal and transverse waves are given separate treatment. Alternately the condition $(\partial A_\mu/\partial x_\mu)\Psi=0$ is carried along as a supplementary condition. In the present form no such special considerations are necessary for we are dealing with the solutions of the equation $-\Box^2 A_\mu=4\pi j_\mu$ with a current j_μ which is conserved $\partial j_\mu/\partial x_\mu=0$. That means at least $\Box^2(\partial A_\mu/\partial x_\mu)=0$ and in fact our solution also satisfies $\partial A_\mu/\partial x_\mu=0$.

To show that this is the case we consider the amplitude for emission (real or virtual) of a photon and show that the divergence of this amplitude vanishes. The amplitude for emission for photons polarized in the μ direction involves matrix elements of γ_μ. Therefore what we have to show is that the corresponding matrix elements of $q_\mu\gamma_\mu=q$ vanish. For example, for a first order effect we would require the matrix element of q between two states p_1 and $p_2=p_1+q$. But since $q=p_2-p_1$ and $(\bar{u}_2 p_1 u_1)=m(\bar{u}_2 u_1)=(\bar{u}_2 p_2 u_1)$ the matrix element vanishes, which proves the contention in this case. It also vanishes in more complex situations (essentially because of relation (34), below) (for example, try putting $e_2=q_2$ in the matrix (15) for the Compton Effect).

To prove this in general, suppose a_i, $i=1$ to N are a set of plane wave disturbing potentials carrying momenta q_i (e.g., some may be emissions or absorptions of the same or different quanta) and consider a matrix for the transition from a state of momentum p_0 to p_N such

[21] E. A. Uehling, Phys. Rev. **48**, 55 (1935), R. Serber, Phys. Rev. **48**, 49 (1935).

[22] There are loops completely without external interactions. For example, a pair is created virtually along with a photon. Next they annihilate, absorbing this photon. Such loops are disregarded on the grounds that they do not interact with anything and are thereby completely unobservable. Any indirect effects they may have via the exclusion principle have already been included.

as $\boldsymbol{a}_N\prod_{i=1}^{N-1}(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i$ where $\boldsymbol{p}_i=\boldsymbol{p}_{i-1}+\boldsymbol{q}_i$ (and in the product, terms with larger i are written to the left). The most general matrix element is simply a linear combination of these. Next consider the matrix between states $\boldsymbol{p}_0$ and $\boldsymbol{p}_N+\boldsymbol{q}$ in a situation in which not only are the $\boldsymbol{a}_i$ acting but also another potential $\boldsymbol{a}\exp(-iq\cdot x)$ where $\boldsymbol{a}=\boldsymbol{q}$. This may act previous to all $\boldsymbol{a}_i$, in which case it gives $\boldsymbol{a}_N\prod(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i(\boldsymbol{p}_0+\boldsymbol{q}-m)^{-1}\boldsymbol{q}$ which is equivalent to $+\boldsymbol{a}_N\prod(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i$ since $+(\boldsymbol{p}_0+\boldsymbol{q}-m)^{-1}\boldsymbol{q}$ is equivalent to $(\boldsymbol{p}_0+\boldsymbol{q}-m)^{-1}\times(\boldsymbol{p}_0+\boldsymbol{q}-m)$ as $\boldsymbol{p}_0$ is equivalent to m acting on the initial state. Likewise if it acts after all the potentials it gives $\boldsymbol{q}(\boldsymbol{p}_N-m)^{-1}\boldsymbol{a}_N\prod(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i$ which is equivalent to $-\boldsymbol{a}_N\prod(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i$ since $\boldsymbol{p}_N+\boldsymbol{q}-m$ gives zero on the final state. Or again it may act between the potential $\boldsymbol{a}_k$ and $\boldsymbol{a}_{k+1}$ for each k. This gives

$$\sum_{k=1}^{N-1}\boldsymbol{a}_N\prod_{i=k+1}^{N-1}(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i(\boldsymbol{p}_k+\boldsymbol{q}-m)^{-1}\times\boldsymbol{q}(\boldsymbol{p}_k-m)^{-1}\boldsymbol{a}_k\prod_{j=1}^{k-1}(\boldsymbol{p}_j-m)^{-1}\boldsymbol{a}_j.$$

However,

$$(\boldsymbol{p}_k+\boldsymbol{q}-m)^{-1}\boldsymbol{q}(\boldsymbol{p}_k-m)^{-1}=(\boldsymbol{p}_k-m)^{-1}-(\boldsymbol{p}_k+\boldsymbol{q}-m)^{-1},\quad(34)$$

so that the sum breaks into the difference of two sums, the first of which may be converted to the other by the replacement of k by $k-1$. There remain only the terms from the ends of the range of summation,

$$+\boldsymbol{a}_N\prod_{i=1}^{N-1}(\boldsymbol{p}_i-m)^{-1}\boldsymbol{a}_i-\boldsymbol{a}_N\prod_{i=1}^{N-1}(\boldsymbol{p}_i+\boldsymbol{q}-m)^{-1}\boldsymbol{a}_i.$$

These cancel the two terms originally discussed so that the entire effect is zero. Hence any wave emitted will satisfy $\partial A_\mu/\partial x_\mu=0$. Likewise longitudinal waves (that is, waves for which $A_\mu=\partial\phi/\partial x_\mu$ or $\boldsymbol{a}=\boldsymbol{q}$) cannot be absorbed and will have no effect, for the matrix elements for emission and absorption are similar. (We have said little more than that a potential $A_\mu=\partial\varphi/\partial x_\mu$ has no effect on a Dirac electron since a transformation $\psi'=\exp(-i\phi)\psi$ removes it. It is also easy to see in coordinate representation using integrations by parts.)

This has a useful practical consequence in that in computing probabilities for transition for unpolarized light one can sum the squared matrix over all four directions rather than just the two special polarization vectors. Thus suppose the matrix element for some process for light polarized in direction e_μ is $e_\mu M_\mu$. If the light has wave vector q_μ we know from the argument above that $q_\mu M_\mu=0$. For unpolarized light progressing in the z direction we would ordinarily calculate $M_x{}^2+M_y{}^2$. But we can as well sum $M_x{}^2+M_y{}^2+M_z{}^2-M_t{}^2$ for $q_\mu M_\mu$ implies $M_t=M_z$ since $q_t=q_z$ for free quanta. This shows that unpolarized light is a relativistically invariant concept, and permits some simplification in computing cross sections for such light.

Incidentally, the virtual quanta interact through terms like $\gamma_\mu\cdots\gamma_\mu\boldsymbol{k}^{-2}d^4k$. Real processes correspond to poles in the formulae for virtual processes. The pole occurs when $\boldsymbol{k}^2=0$, but it looks at first as though in the sum on all four values of μ, of $\gamma_\mu\cdots\gamma_\mu$ we would have four kinds of polarization instead of two. Now it is clear that only two perpendicular to $\boldsymbol{k}$ are effective.

The usual elimination of longitudinal and scalar virtual photons (leading to an instantaneous Coulomb potential) can of course be performed here too (although it is not particularly useful). A typical term in a virtual transition is $\gamma_\mu\cdots\gamma_\mu\boldsymbol{k}^{-2}d^4k$ where the $\cdots$ represent some intervening matrices. Let us choose for the values of μ, the time t, the direction of vector part $\mathbf{K}$, of $\boldsymbol{k}$, and two perpendicular directions 1, 2. We shall not change the expression for these two 1, 2 for these are represented by transverse quanta. But we must find $(\gamma_t\cdots\gamma_t)-(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})$. Now $\boldsymbol{k}=k_4\gamma_t-K\gamma_{\mathbf{K}}$, where $K=(\mathbf{K}\cdot\mathbf{K})^{\frac{1}{2}}$, and we have shown above that $\boldsymbol{k}$ replacing the γ_μ gives zero.[23] Hence $K\gamma_{\mathbf{K}}$ is equivalent to $k_4\gamma_t$ and

$$(\gamma_t\cdots\gamma_t)-(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})=((K^2-k_4{}^2)/K^2)(\gamma_t\cdots\gamma_t),$$

so that on multiplying by $\boldsymbol{k}^{-2}d^4k=d^4k(k_4{}^2-K^2)^{-1}$ the net effect is $-(\gamma_t\cdots\gamma_t)d^4k/K^2$. The γ_t means just scalar waves, that is, potentials produced by charge density. The fact that $1/K^2$ does not contain k_4 means that k_4 can be integrated first, resulting in an instantaneous interaction, and the $d^3\mathbf{K}/K^2$ is just the momentum representation of the Coulomb potential, $1/r$.

9. KLEIN GORDON EQUATION

The methods may be readily extended to particles of spin zero satisfying the Klein Gordon equation,[24]

$$\Box^2\psi-m^2\psi=i\partial(A_\mu\psi)/\partial x_\mu+iA_\mu\partial\psi/\partial x_\mu-A_\mu A_\mu\psi.\quad(35)$$

[23] A little more care is required when both γ_μ's act on the same particle. Define $\boldsymbol{x}=k_4\gamma_t+K\gamma_{\mathbf{K}}$, and consider $(\boldsymbol{k}\cdots\boldsymbol{x})+(\boldsymbol{x}\cdots\boldsymbol{k})$. Exactly this term would arise if a system, acted on by potential $\boldsymbol{x}$ carrying momentum $-\boldsymbol{k}$, is disturbed by an added potential $\boldsymbol{k}$ of momentum $+\boldsymbol{k}$ (the reversed sign of the momenta in the intermediate factors in the second term $\boldsymbol{x}\cdots\boldsymbol{k}$ has no effect since we will later integrate over all $\boldsymbol{k}$). Hence as shown above the result is zero, but since $(\boldsymbol{k}\cdots\boldsymbol{x})+(\boldsymbol{x}\cdots\boldsymbol{k})=k_4{}^2(\gamma_t\cdots\gamma_t)-K^2(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})$ we can still conclude $(\gamma_{\mathbf{K}}\cdots\gamma_{\mathbf{K}})=k_4{}^2K^{-2}(\gamma_t\cdots\gamma_t)$.

[24] The equations discussed in this section were deduced from the formulation of the Klein Gordon equation given in reference 5, Section 14. The function ψ in this section has only one component and is not a spinor. An alternative formal method of making the equations valid for spin zero and also for spin 1 is (presumably) by use of the Kemmer-Duffin matrices β_μ, satisfying the commutation relation

$$\beta_\mu\beta_\nu\beta_\sigma+\beta_\sigma\beta_\nu\beta_\mu=\delta_{\mu\nu}\beta_\sigma+\delta_{\sigma\nu}\beta_\mu.$$

If we interpret $\boldsymbol{a}$ to mean $a_\mu\beta_\mu$, rather than $a_\mu\gamma_\mu$, for any a_μ, all of the equations in momentum space will remain formally identical to those for the spin 1/2; with the exception of those in which a denominator $(\boldsymbol{p}-m)^{-1}$ has been rationalized to $(\boldsymbol{p}+m)(\boldsymbol{p}^2-m^2)^{-1}$ since $\boldsymbol{p}^2$ is no longer equal to a number, $p\cdot p$. But $\boldsymbol{p}^3$ does equal $(p\cdot p)\boldsymbol{p}$ so that $(\boldsymbol{p}-m)^{-1}$ may now be interpreted as $(m\boldsymbol{p}+m^2+\boldsymbol{p}^2-p\cdot p)(p\cdot p-m^2)^{-1}m^{-1}$. This implies that equations in coordinate space will be valid of the function $K_+(2,1)$ is given as $K_+(2,1)=[(i\nabla_2+m)-m^{-1}(\nabla_2{}^2+\Box_2{}^2)]iI_+(2,1)$ with $\nabla_2=\beta_\mu\partial/\partial x_{2\mu}$. This is all in virtue of the fact that the many component wave function ψ (5 components for spin 0, 10 for spin 1) satisfies $(i\nabla-m)\psi=\boldsymbol{A}\psi$ which is formally identical to the Dirac Equation. See W. Pauli, Rev. Mod. Phys. **13**, 203 (1940).

The important kernel is now $I_+(2, 1)$ defined in (I, Eq. (32)). For a free particle, the wave function $\psi(2)$ satisfies $+\Box^2\psi - m^2\psi = 0$. At a point, 2, inside a space time region it is given by

$$\psi(2) = \int [\psi(1)\partial I_+(2, 1)/\partial x_{1\mu} - (\partial\psi/\partial x_{1\mu}) I_+(2, 1)] N_\mu(1) d^3V_1,$$

(as is readily shown by the usual method of demonstrating Green's theorem) the integral being over an entire 3-surface boundary of the region (with normal vector N_μ). Only the positive frequency components of ψ contribute from the surface preceding the time corresponding to 2, and only negative frequencies from the surface future to 2. These can be interpreted as electrons and positrons in direct analogy to the Dirac case.

The right-hand side of (35) can be considered as a source of new waves and a series of terms written down to represent matrix elements for processes of increasing order. There is only one new point here, the term in $A_\mu A_\mu$ by which two quanta can act at the same time. As an example, suppose three quanta or potentials, $a_\mu \exp(-iq_a\cdot x)$, $b_\mu \exp(-iq_b\cdot x)$, and $c_\mu \exp(-iq_c\cdot x)$ are to act in that order on a particle of original momentum $p_{0\mu}$ so that $\boldsymbol{p}_a = \boldsymbol{p}_0 + \boldsymbol{q}_a$ and $\boldsymbol{p}_b = \boldsymbol{p}_a + \boldsymbol{q}_b$; the final momentum being $\boldsymbol{p}_c = \boldsymbol{p}_b + \boldsymbol{q}_c$. The matrix element is the sum of three terms ($\boldsymbol{p}^2 = p_\mu p_\mu$) (illustrated in Fig. 7)

$$\begin{aligned}&(p_c\cdot c + p_b\cdot c)(\boldsymbol{p}_b^2 - m^2)^{-1}(p_b\cdot b + p_a\cdot b)\\ &\qquad\times(\boldsymbol{p}_a^2 - m^2)^{-1}(p_a\cdot a + p_0\cdot a)\\ &-(p_c\cdot c + p_b\cdot c)(\boldsymbol{p}_b^2 - m^2)^{-1}(b\cdot a)\\ &-(c\cdot b)(\boldsymbol{p}_a^2 - m^2)^{-1}(p_a\cdot a + p_0\cdot a).\end{aligned} \tag{36}$$

The first comes when each potential acts through the perturbation $i\partial(A_\mu\psi)/\partial x_\mu + iA_\mu\partial\psi/\partial x_\mu$. These gradient operators in momentum space mean respectively the momentum after and before the potential A_μ operates. The second term comes from b_μ and a_μ acting at the same instant and arises from the $A_\mu A_\mu$ term in (a). Together b_μ and a_μ carry momentum $q_{b\mu} + q_{a\mu}$ so that after $b\cdot a$ operates the momentum is $\boldsymbol{p}_0 + \boldsymbol{q}_a + \boldsymbol{q}_b$ or $\boldsymbol{p}_b$. The final term comes from c_μ and b_μ operating together in a similar manner. The term $A_\mu A_\mu$ thus permits a new type of process in which two quanta can be emitted (or absorbed, or one absorbed, one emitted) at the same time. There is no $a\cdot c$ term for the order a, b, c we have assumed. In an actual problem there would be other terms like (36) but with alterations in the order in which the quanta a, b, c act. In these terms $a\cdot c$ would appear.

As a further example the self-energy of a particle of momentum p_μ is

$$(e^2/2\pi im)\int [(2p-k)_\mu((\boldsymbol{p}-\boldsymbol{k})^2 - m^2)^{-1} \times (2p-k)_\mu - \delta_{\mu\mu}] d^4k \boldsymbol{k}^{-2} C(\boldsymbol{k}^2),$$

where the $\delta_{\mu\mu} = 4$ comes from the $A_\mu A_\mu$ term and represents the possibility of the simultaneous emission and absorption of the same virtual quantum. This integral without the $C(\boldsymbol{k}^2)$ diverges quadratically and would not converge if $C(\boldsymbol{k}^2) = -\lambda^2/(\boldsymbol{k}^2 - \lambda^2)$. Since the interaction occurs through the gradients of the potential, we must use a stronger convergence factor, for example $C(\boldsymbol{k}^2) = \lambda^4(\boldsymbol{k}^2 - \lambda^2)^{-2}$, or in general (17) with $\int_0^\infty \lambda^2 G(\lambda) d\lambda = 0$. In this case the self-energy converges but depends quadratically on the cut-off λ and is not necessarily small compared to m. The radiative corrections to scattering after mass renormalization are insensitive to the cut-off just as for the Dirac equation.

When there are several particles one can obtain Bose statistics by the rule that if two processes lead to the same state but with two electrons exchanged, their amplitudes are to be added (rather than subtracted as for Fermi statistics). In this case equivalence to the second quantization treatment of Pauli and Weisskopf should be demonstrable in a way very much like that given in I (appendix) for Dirac electrons. The Bose statistics mean that the sign of contribution of a closed loop to the vacuum polarization is the opposite of what it is for the Fermi case (see I). It is ($\boldsymbol{p}_b = \boldsymbol{p}_a + \boldsymbol{q}$)

$$J_{\mu\nu} = \frac{e^2}{2\pi im}\int [(p_{b\mu} + p_{a\mu})(p_{b\nu} + p_{a\nu})(\boldsymbol{p}_a^2 - m^2)^{-1} \times (\boldsymbol{p}_b^2 - m^2)^{-1} - \delta_{\mu\nu}(\boldsymbol{p}_a^2 - m^2)^{-1} - \delta_{\mu\nu}(\boldsymbol{p}_b^2 - m^2)^{-1}] d^4p_a$$

giving,

$$J_{\mu\nu}{}^P = \frac{e^2}{\pi}(q_\mu q_\nu - \delta_{\mu\nu}\boldsymbol{q}^2)\left[\frac{1}{6}\ln\frac{\lambda^2}{m^2} + \frac{1}{9} - \frac{4m^2 - \boldsymbol{q}^2}{3\boldsymbol{q}^2}\left(1 - \frac{\theta}{\tan\theta}\right)\right],$$

the notation as in (33). The imaginary part for $(\boldsymbol{q}^2)^{\frac{1}{2}} > 2m$ is again positive representing the loss in the probability of finding the final state to be a vacuum, associated with the possibilities of pair production. Fermi statistics would give a gain in probability (and also a charge renormalization of opposite sign to that expected).

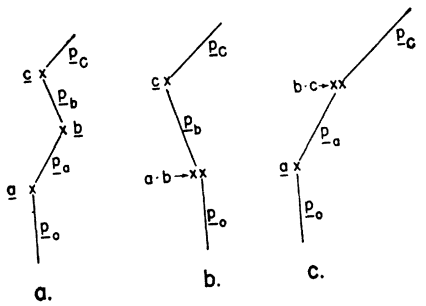

FIG. 7. Klein-Gordon particle in three potentials, Eq. (36). The coupling to the electromagnetic field is now, for example, $p_0\cdot a + p_a\cdot a$, and a new possibility arises, (b), of simultaneous interaction with two quanta $a\cdot b$. The propagation factor is now $(p\cdot p - m^2)^{-1}$ for a particle of momentum p_μ.

10. APPLICATION TO MESON THEORIES

The theories which have been developed to describe mesons and the interaction of nucleons can be easily expressed in the language used here. Calculations, to lowest order in the interactions can be made very easily for the various theories, but agreement with experimental results is not obtained. Most likely all of our present formulations are quantitatively unsatisfactory. We shall content ourselves therefore with a brief summary of the methods which can be used.

The nucleons are usually assumed to satisfy Dirac's equation so that the factor for propagation of a nucleon of momentum $\boldsymbol{p}$ is $(\boldsymbol{p}-M)^{-1}$ where M is the mass of the nucleon (which implies that nucleons can be created in pairs). The nucleon is then assumed to interact with mesons, the various theories differing in the form assumed for this interaction.

First, we consider the case of neutral mesons. The theory closest to electrodynamics is the theory of vector mesons with vector coupling. Here the factor for emission or absorption of a meson is $g\gamma_\mu$ when this meson is "polarized" in the μ direction. The factor g, the "mesonic charge," replaces the electric charge e. The amplitude for propagation of a meson of momentum $\boldsymbol{q}$ in intermediate states is $(\boldsymbol{q}^2-\mu^2)^{-1}$ (rather than $\boldsymbol{q}^{-2}$ as it is for light) where μ is the mass of the meson. The necessary integrals are made finite by convergence factors $C(\boldsymbol{q}^2-\mu^2)$ as in electrodynamics. For scalar mesons with scalar coupling the only change is that one replaces the γ_μ by 1 in emission and absorption. There is no longer a direction of polarization, μ, to sum upon. For pseudoscalar mesons, pseudoscalar coupling replace γ_μ by $\gamma_5=i\gamma_x\gamma_y\gamma_z\gamma_t$. For example, the self-energy matrix of a nucleon of momentum $\boldsymbol{p}$ in this theory is

$$(g^2/\pi i)\int \gamma_5(\boldsymbol{p}-\boldsymbol{k}-M)^{-1}\gamma_5 d^4k(\boldsymbol{k}^2-\mu^2)^{-1}C(\boldsymbol{k}^2-\mu^2).$$

Other types of meson theory result from the replacement of γ_μ by other expressions (for example by $\frac{1}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$ with a subsequent sum over all μ and ν for virtual mesons). Scalar mesons with vector coupling result from the replacement of γ_μ by $\mu^{-1}\boldsymbol{q}$ where $\boldsymbol{q}$ is the final momentum of the nucleon minus its initial momentum, that is, it is the momentum of the meson if absorbed, or the negative of the momentum of a meson emitted. As is well known, this theory with neutral mesons gives zero for all processes, as is proved by our discussion on longitudinal waves in electrodynamics. Pseudoscalar mesons with pseudo-vector coupling corresponds to γ_μ being replaced by $\mu^{-1}\gamma_5\boldsymbol{q}$ while vector mesons with tensor coupling correspond to using $(2\mu)^{-1}(\gamma_\mu\boldsymbol{q}-\boldsymbol{q}\gamma_\mu)$. These extra gradients involve the danger of producing higher divergencies for real processes. For example, $\gamma_5\boldsymbol{q}$ gives a logarithmically divergent interaction of neutron and electron.[25] Although these divergencies can be held by strong enough convergence factors, the results then are sensitive to the method used for convergence and the size of the cut-off values of λ. For low order processes $\mu^{-1}\gamma_5\boldsymbol{q}$ is equivalent to the pseudoscalar interaction $2M\mu^{-1}\gamma_5$ because if taken between free particle wave functions of the nucleon of momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q}$, we have

$$(\bar{u}_2\gamma_5\boldsymbol{q}u_1)=(\bar{u}_2\gamma_5(\boldsymbol{p}_2-\boldsymbol{p}_1)u_1)=-(\bar{u}_2\boldsymbol{p}_2\gamma_5u_1)-(\bar{u}_2\gamma_5\boldsymbol{p}_1u_1)=-2M(\bar{u}_2\gamma_5u_1)$$

since γ_5 anticommutes with $\boldsymbol{p}_2$ and $\boldsymbol{p}_2$ operating on the state 2 equivalent to M as is $\boldsymbol{p}_1$ on the state 1. This shows that the γ_5 interaction is unusually weak in the non-relativistic limit (for example the expected value of γ_5 for a free nucleon is zero), but since $\gamma_5^2=1$ is not small, pseudoscalar theory gives a more important interaction in second order than it does in first. Thus the pseudoscalar coupling constant should be chosen to fit nuclear forces including these important second order processes.[26] The equivalence of pseudoscalar and pseudovector coupling which holds for low order processes therefore does not hold when the pseudoscalar theory is giving its most important effects. These theories will therefore give quite different results in the majority of practical problems.

In calculating the corrections to scattering of a nucleon by a neutral vector meson field (γ_μ) due to the effects of virtual mesons, the situation is just as in electrodynamics, in that the result converges without need for a cut-off and depends only on gradients of the meson potential. With scalar (1) or pseudoscalar (γ_5) neutral mesons the result diverges logarithmically and so must be cut off. The part sensitive to the cut-off, however, is directly proportional to the meson potential. It may thereby be removed by a renormalization of mesonic charge g. After this renormalization the results depend only on gradients of the meson potential and are essentially independent of cut-off. This is in addition to the mesonic charge renormalization coming from the production of virtual nucleon pairs by a meson, analogous to the vacuum polarization in electrodynamics. But here there is a further difference from electrodynamics for scalar or pseudoscalar mesons in that the polarization also gives a term in the induced current proportional to the meson potential representing therefore an additional renormalization of the *mass of the meson* which usually depends quadratically on the cut-off.

Next consider charged mesons in the absence of an electromagnetic field. One can introduce isotopic spin operators in an obvious way. (Specifically replace the neutral γ_5, say, by $\tau_i\gamma_5$ and sum over $i=1, 2$ where $\tau_1=\tau_++\tau_-$, $\tau_2=i(\tau_+-\tau_-)$ and τ_+ changes neutron to proton (τ_+ on proton$=0$) and τ_- changes proton to neutron.) It is just as easy for practical problems simply to keep track of whether the particle is a proton or a neutron on a diagram drawn to help write down the

[25] M. Slotnick and W. Heitler, Phys. Rev. **75**, 1645 (1949).

[26] H. A. Bethe, Bull. Am. Phys. Soc. **24**, 3, Z3 (Washington, 1949).

matrix element. This excludes certain processes. For example in the scattering of a negative meson from $\boldsymbol{q}_1$ to $\boldsymbol{q}_2$ by a neutron, the meson $\boldsymbol{q}_2$ must be emitted first (in order of operators, not time) for the neutron cannot absorb the negative meson $\boldsymbol{q}_1$ until it becomes a proton. That is, in comparison to the Klein Nishina formula (15), only the analogue of second term (see Fig. 5(b)) would appear in the scattering of negative mesons by neutrons, and only the first term (Fig. 5(a)) in the neutron scattering of positive mesons.

The source of mesons of a given charge is not conserved, for a neutron capable of emitting negative mesons may (on emitting one, say) become a proton no longer able to do so. The proof that a perturbation $\boldsymbol{q}$ gives zero, discussed for longitudinal electromagnetic waves, fails. This has the consequence that vector mesons, if represented by the interaction γ_μ would not satisfy the condition that the divergence of the potential is zero. The interaction is to be taken[27] as $\gamma_\mu - \mu^{-2}q_\mu\boldsymbol{q}$ in emission and as γ_μ in absorption if the real emission of mesons with a non-zero divergence of potential is to be avoided. (The correction term $\mu^{-2}q_\mu\boldsymbol{q}$ gives zero in the neutral case.) The asymmetry in emission and absorption is only apparent, as this is clearly the same thing as subtracting from the original $\gamma_\mu\cdots\gamma_\mu$, a term $\mu^{-2}\boldsymbol{q}\cdots\boldsymbol{q}$. That is, if the term $-\mu^{-2}q_\mu\boldsymbol{q}$ is omitted the resulting theory describes a combination of mesons of spin one and spin zero. The spin zero mesons, coupled by vector coupling $\boldsymbol{q}$, are removed by subtracting the term $\mu^{-2}\boldsymbol{q}\cdots\boldsymbol{q}$.

The two extra gradients $\boldsymbol{q}\cdots\boldsymbol{q}$ make the problem of diverging integrals still more serious (for example the interaction between two protons corresponding to the exchange of two charged vector mesons depends quadratically on the cut-off if calculated in a straightforward way). One is tempted in this formulation to choose simply $\gamma_\mu\cdots\gamma_\mu$ and accept the admixture of spin zero mesons. But it appears that this leads in the conventional formalism to negative energies for the spin zero component. This shows one of the advantages of the method of second quantization of meson fields over the present formulation. There such errors of sign are obvious while here we seem to be able to write seemingly innocent expressions which can give absurd results. Pseudovector mesons with pseudovector coupling correspond to using $\gamma_5(\gamma_\mu - \mu^{-2}q_\mu\boldsymbol{q})$ for absorption and $\gamma_5\gamma_\mu$ for emission for both charged and neutral mesons.

In the presence of an electromagnetic field, whenever the nucleon is a proton it interacts with the field in the way described for electrons. The meson interacts in the scalar or pseudoscalar case as a particle obeying the Klein-Gordon equation. It is important here to use the method of calculation of Bethe and Pauli, that is, a virtual meson is assumed to have the same "mass" during all its interactions with the electromagnetic field. The result for mass μ and for $(\mu^2+\lambda^2)^{\frac{1}{2}}$ are subtracted and the difference integrated over the function $G(\lambda)d\lambda$. A separate convergence factor is not provided for each meson propagation between electromagnetic interactions, otherwise gauge invariance is not insured. When the coupling involves a gradient, such as $\gamma_5\boldsymbol{q}$ where $\boldsymbol{q}$ is the final minus the initial momentum of the nucleon, the vector potential $\boldsymbol{A}$ must be subtracted from the momentum of the proton. That is, there is an additional coupling $\pm\gamma_5\boldsymbol{A}$ (plus when going from proton to neutron, minus for the reverse) representing the new possibility of a simultaneous emission (or absorption) of meson and photon.

Emission of positive or absorption of negative virtual mesons are represented in the same term, the sign of the charge being determined by temporal relations as for electrons and positrons.

Calculations are very easily carried out in this way to lowest order in g^2 for the various theories for nucleon interaction, scattering of mesons by nucleons, meson production by nuclear collisions and by gamma-rays, nuclear magnetic moments, neutron electron scattering, etc., However, no good agreement with experiment results, when these are available, is obtained. Probably all of the formulations are incorrect. An uncertainty arises since the calculations are only to first order in g^2, and are not valid if $g^2/\hbar c$ is large.

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APPENDIX

In this appendix a method will be illustrated by which the simpler integrals appearing in problems in electrodynamics can be directly evaluated. The integrals arising in more complex processes lead to rather complicated functions, but the study of the relations of one integral to another and their expression in terms of simpler integrals may be facilitated by the methods given here.

[27] The vector meson field potentials φ_μ satisfy

$$-\partial/\partial x_\nu(\partial\varphi_\mu/\partial x_\nu - \partial\varphi_\nu/\partial x_\mu) - \mu^2\varphi_\mu = -4\pi s_\mu,$$

where s_μ, the source for such mesons, is the matrix element of γ_μ between states of neutron and proton. By taking the divergence $\partial/\partial x_\mu$ of both sides, conclude that $\partial\varphi_\nu/\partial x_\nu = 4\pi\mu^{-2}\partial s_\nu/\partial x_\nu$ so that the original equation can be rewritten as

$$\Box^2\varphi_\mu - \mu^2\varphi_\mu = -4\pi(s_\mu + \mu^{-2}\partial/\partial x_\mu(\partial s_\nu/\partial x_\nu)).$$

The right hand side gives in momentum representation $\gamma_\mu - \mu^{-2}q_\mu q_\nu\gamma_\nu$ the left yields the $(q^2-\mu^2)^{-1}$ and finally the interaction $s_\mu\varphi_\mu$ in the Lagrangian gives the γ_μ on absorption.

Proceeding in this way find generally that particles of spin one can be represented by a four-vector u_μ (which, for a free particle of momentum q satisfies $q\cdot u=0$). The propagation of virtual particles of momentum q from state ν to μ is represented by multiplication by the 4–4 matrix (or tensor) $P_{\mu\nu}=(\delta_{\mu\nu}-\mu^{-2}q_\mu q_\nu)\times(q^2-\mu^2)^{-1}$. The first-order interaction (from the Proca equation) with an electromagnetic potential $a\exp(-ik\cdot x)$ corresponds to multiplication by the matrix $E_{\mu\nu}=(q_2\cdot a+q_1\cdot a)\delta_{\mu\nu}-q_{2\nu}a_\mu-q_{1\mu}a_\nu$ where q_1 and $q_2=q_1+k$ are the momenta before and after the interaction. Finally, two potentials a, b may act simultaneously, with matrix $E'_{\mu\nu}=-(a\cdot b)\delta_{\mu\nu}+b_\mu a_\nu$.

As a typical problem consider the integral (12) appearing in the first order radiationless scattering problem:

$$\int \gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}-m)^{-1}\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}-m)^{-1}\gamma_\mu \boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2), \tag{1a}$$

where we shall take $C(\boldsymbol{k}^2)$ to be typically $-\lambda^2(\boldsymbol{k}^2-\lambda^2)^{-1}$ and d^4k means $(2\pi)^{-2}dk_1dk_2dk_3dk_4$. We first rationalize the factors $(\boldsymbol{p}-\boldsymbol{k}-m)^{-1}=(\boldsymbol{p}-\boldsymbol{k}+m)((\boldsymbol{p}-\boldsymbol{k})^2-m^2)^{-1}$ obtaining,

$$\int \gamma_\mu(\boldsymbol{p}_2-\boldsymbol{k}+m)\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}+m)\gamma_\mu \boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2) \times((\boldsymbol{p}_1-\boldsymbol{k})^2-m^2)^{-1}((\boldsymbol{p}_2-\boldsymbol{k})^2-m^2)^{-1}. \tag{2a}$$

The matrix expression may be simplified. It appears to be best to do so *after* the integrations are performed. Since $\boldsymbol{AB}=2A\cdot B-\boldsymbol{BA}$ where $A\cdot B=A_\mu B_\mu$ is a number commuting with all matrices, find, if R is any expression, and $\boldsymbol{A}$ a vector, since $\gamma_\mu \boldsymbol{A}=-\boldsymbol{A}\gamma_\mu+2A_\mu$,

$$\gamma_\mu \boldsymbol{A}R\gamma_\mu=-\boldsymbol{A}\gamma_\mu R\gamma_\mu+2R\boldsymbol{A}. \tag{3a}$$

Expressions between two γ_μ's can be thereby reduced by induction. Particularly useful are

$$\begin{aligned}\gamma_\mu\gamma_\mu&=4\\ \gamma_\mu \boldsymbol{A}\gamma_\mu&=-2\boldsymbol{A}\\ \gamma_\mu \boldsymbol{AB}\gamma_\mu&=2(\boldsymbol{AB}+\boldsymbol{BA})=4A\cdot B\\ \gamma_\mu \boldsymbol{ABC}\gamma_\mu&=-2\boldsymbol{CBA}\end{aligned} \tag{4a}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are any three vector-matrices (i.e., linear combinations of the four γ's).

In order to calculate the integral in (2a) the integral may be written as the sum of three terms (since $\boldsymbol{k}=k_\sigma\gamma_\sigma$),

$$\gamma_\mu(\boldsymbol{p}_2+m)\boldsymbol{a}(\boldsymbol{p}_1+m)\gamma_\mu J_1-[\gamma_\mu\gamma_\sigma \boldsymbol{a}(\boldsymbol{p}_1+m)\gamma_\mu +\gamma_\mu(\boldsymbol{p}_2+m)\boldsymbol{a}\gamma_\sigma\gamma_\mu]J_2+\gamma_\mu\gamma_\sigma\boldsymbol{a}\gamma_\tau\gamma_\mu J_3, \tag{5a}$$

where

$$J_{(1;2;3)}=\int (1;k_\sigma;k_\sigma k_\tau)\boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2) \times((\boldsymbol{p}_2-\boldsymbol{k})^2-m^2)^{-1}((\boldsymbol{p}_1-\boldsymbol{k})^2-m^2)^{-1}. \tag{6a}$$

That is for J_1 the $(1;k_\sigma;k_\sigma k_\tau)$ is replaced by 1, for J_2 by k_σ, and for J_3 by $k_\sigma k_\tau$.

More complex processes of the first order involve more factors like $((\boldsymbol{p}_3-\boldsymbol{k})^2-m^2)^{-1}$ and a corresponding increase in the number of k's which may appear in the numerator, as $k_\sigma k_\tau k_\nu\cdots$. Higher order processes involving two or more virtual quanta involve similar integrals but with factors possibly involving $\boldsymbol{k}+\boldsymbol{k}'$ instead of just $\boldsymbol{k}$, and the integral extending on $\boldsymbol{k}^{-2}d^4kC(\boldsymbol{k}^2)\boldsymbol{k}'^{-2}d^4k'C(\boldsymbol{k}'^2)$. They can be simplified by methods analogous to those used on the first order integrals.

The factors $(\boldsymbol{p}-\boldsymbol{k})^2-m^2$ may be written

$$(\boldsymbol{p}-\boldsymbol{k})^2-m^2=\boldsymbol{k}^2-2p\cdot k-\Delta, \tag{7a}$$

where $\Delta=m^2-\boldsymbol{p}^2$, $\Delta_1=m_1{}^2-\boldsymbol{p}_1{}^2$, etc., and we can consider dealing with cases of greater generality in that the different denominators need not have the same value of the mass m. In our specific problem (6a), $\boldsymbol{p}_1{}^2=m^2$ so that $\Delta_1=0$, but we desire to work with greater generality.

Now for the factor $C(\boldsymbol{k}^2)/\boldsymbol{k}^2$ we shall use $-\lambda^2(\boldsymbol{k}^2-\lambda^2)^{-1}\boldsymbol{k}^{-2}$. This can be written as

$$-\lambda^2/(\boldsymbol{k}^2-\lambda^2)\boldsymbol{k}^2=\boldsymbol{k}^{-2}C(\boldsymbol{k}^2)=-\int_0^{\lambda^2} dL(\boldsymbol{k}^2-L)^{-2}. \tag{8a}$$

Thus we can replace $\boldsymbol{k}^{-2}C(\boldsymbol{k}^2)$ by $(\boldsymbol{k}^2-L)^{-2}$ and at the end integrate the result with respect to L from zero to λ^2. We can for many practical purposes consider λ^2 very large relative to m^2 or $\boldsymbol{p}^2$. When the original integral converges even without the convergence factor, it will be obvious since the L integration will then be convergent to infinity. If an infra-red catastrophe exists in the integral one can simply assume quanta have a small mass $\lambda_{\min}$ and extend the integral on L from $\lambda^2{}_{\min}$ to λ^2, rather than from zero to λ^2.

We then have to do integrals of the form

$$\int (1;k_\sigma;k_\sigma k_\tau)d^4k(\boldsymbol{k}^2-L)^{-2}(\boldsymbol{k}^2-2p_1\cdot k-\Delta_1)^{-1} \times(\boldsymbol{k}^2-2p_2\cdot k-\Delta_2)^{-1}, \tag{9a}$$

where by $(1;k_\sigma;\cdot k_\sigma k_\tau)$ we mean that in the place of this symbol either 1, or k_σ, or $k_\sigma k_\tau$ may stand in different cases. In more complicated problems there may be more factors $(\boldsymbol{k}^2-2p_i\cdot k-\Delta_i)^{-1}$ or other powers of these factors (the $(\boldsymbol{k}^2-L)^{-2}$ may be considered as a special case of such a factor with $\boldsymbol{p}_i=0$, $\Delta_i=L$) and further factors like $k_\sigma k_\tau k_\rho\cdots$ in the numerator. The poles in all the factors are made definite by the assumption that L, and the Δ's have infinitesimal negative imaginary parts.

We shall do the integrals of successive complexity by induction. We start with the simplest convergent one, and show

$$\int d^4k(\boldsymbol{k}^2-L)^{-3}=(8iL)^{-1}. \tag{10a}$$

For this integral is $\int(2\pi)^{-2}dk_4d^3\mathbf{K}(k_4{}^2-\mathbf{K}\cdot\mathbf{K}-L)^{-3}$ where the vector $\mathbf{K}$, of magnitude $K=(\mathbf{K}\cdot\mathbf{K})^{\frac{1}{2}}$ is k_1, k_2, k_3. The integral on k_4 shows third order poles at $k_4=+(K^2+L)^{\frac{1}{2}}$ and $k_4=-(K^2+L)^{\frac{1}{2}}$. Imagining, in accordance with our definitions, that L has a small negative imaginary part only the first is below the real axis. The contour can be closed by an infinite semi-circle below this axis, without change of the value of the integral since the contribution from the semi-circle vanishes in the limit. Thus the contour can be shrunk about the pole $k_4=+(K^2+L)^{\frac{1}{2}}$ and the resulting k_4 integral is $-2\pi i$ times the residue at this pole. Writing $k_4=(K^2+L)^{\frac{1}{2}}+\epsilon$ and expanding $(k_4{}^2-K^2-L)^{-3}=\epsilon^{-3}(\epsilon+2(K^2+L)^{\frac{1}{2}})^{-3}$ in powers of ϵ, the residue, being the coefficient of the term ϵ^{-1}, is seen to be $6(2(K^2+L)^{\frac{1}{2}})^{-5}$ so our integral is

$$-(3i/32\pi)\int_0^\infty 4\pi K^2dK(K^2+L)^{-5/2}=(3/8i)(1/3L)$$

establishing (10a).

We also have $\int k_\sigma d^4k(\boldsymbol{k}^2-L)^{-3}=0$ from the symmetry in the k space. We write these results as

$$(8i)\int (1;k_\sigma)d^4k(\boldsymbol{k}^2-L)^{-3}=(1;0)L^{-1}, \tag{11a}$$

where in the brackets $(1;k_\sigma)$ and $(1;0)$ corresponding entries are to be used.

Substituting $\boldsymbol{k}=\boldsymbol{k}'-\boldsymbol{p}$ in (11a), and calling $L-\boldsymbol{p}^2=\Delta$ shows that

$$(8i)\int (1;k_\sigma)d^4k(\boldsymbol{k}^2-2p\cdot k-\Delta)^{-3}=(1;p_\sigma)(\boldsymbol{p}^2+\Delta)^{-1}. \tag{12a}$$

By differentiating both sides of (12a) with respect to Δ, or with respect to p_τ there follows directly

$$(24i)\int (1;k_\sigma;k_\sigma k_\tau)d^4k(\boldsymbol{k}^2-2p\cdot k-\Delta)^{-4} =-(1;p_\sigma;p_\sigma p_\tau-\tfrac{1}{2}\delta_{\sigma\tau}(\boldsymbol{p}^2+\Delta))(\boldsymbol{p}^2+\Delta)^{-2}. \tag{13a}$$

Further differentiations give directly successive integrals including more k factors in the numerator and higher powers of $(\boldsymbol{k}^2-2p\cdot k-\Delta)$ in the denominator.

The integrals so far only contain one factor in the denominator. To obtain results for two factors we make use of the identity

$$a^{-1}b^{-1}=\int_0^1 dx(ax+b(1-x))^{-2}, \tag{14a}$$

(suggested by some work of Schwinger's involving Gaussian integrals). This represents the product of two reciprocals as a parametric integral over one and will therefore permit integrals with two factors to be expressed in terms of one. For other powers of a, b, we make use of all of the identities, such as

$$a^{-2}b^{-1}=\int_0^1 2xdx(ax+b(1-x))^{-3}, \tag{15a}$$

deducible from (14a) by successive differentiations with respect to a or b.

To perform an integral, such as

$$(8i)\int (1;k_\sigma)d^4k(\boldsymbol{k}^2-2p_1\cdot k-\Delta_1)^{-2}(\boldsymbol{k}^2-2p_2\cdot k-\Delta_2)^{-1}, \tag{16a}$$

write, using (15a),

$$(k^2-2p_1\cdot k-\Delta_1)^{-2}(k^2-2p_2\cdot k-\Delta_2)^{-1}=\int_0^1 2xdx(k^2-2p_x\cdot k-\Delta_x)^{-3},$$

where

$$p_x=xp_1+(1-x)p_2 \quad \text{and} \quad \Delta_x=x\Delta_1+(1-x)\Delta_2, \tag{17a}$$

(note that Δ_x is *not* equal to $m^2-p_x^2$) so that the expression (16a) is $(8i)\int_0^1 2xdx\int(1;k_\sigma)d^4k(k^2-2p_x\cdot k-\Delta_x)^{-3}$ which may now be evaluated by (12a) and is

$$(16a)=\int_0^1(1;p_{x\sigma})2xdx(p_x^2+\Delta_x)^{-1}, \tag{18a}$$

where p_x, Δ_x are given in (17a). The integral in (18a) is elementary, being the integral of ratio of polynomials, the denominator of second degree in x. The general expression although readily obtained is a rather complicated combination of roots and logarithms.

Other integrals can be obtained again by parametric differentiation. For example differentiation of (16a), (18a) with respect to Δ_2 or $p_{2\tau}$ gives

$$(8i)\int(1;k_\sigma;k_\sigma k_\tau)d^4k(k^2-2p_1\cdot k-\Delta_1)^{-2}(k^2-2p_2\cdot k-\Delta_2)^{-2}$$
$$=-\int_0^1(1;p_{x\sigma};p_{x\sigma}p_{x\tau}-\tfrac{1}{2}\delta_{\sigma\tau}(p_x^2+\Delta_x))\times 2x(1-x)dx(p_x^2+\Delta_x)^{-2}, \tag{19a}$$

again leading to elementary integrals.

As an example, consider the case that the second factor is just $(k^2-L)^{-2}$ and in the first put $p_1=p$, $\Delta_1=\Delta$. Then $p_x=xp$, $\Delta_x=x\Delta+(1-x)L$. There results

$$(8i)\int(1;k_\sigma;k_\sigma k_\tau)d^4k(k^2-L)^{-2}(k^2-2p\cdot k-\Delta)^{-2}$$
$$=-\int_0^1(1;xp_\sigma;x^2p_\sigma p_\tau-\tfrac{1}{2}\delta_{\sigma\tau}(x^2p^2+\Delta_x))\times 2x(1-x)dx(x^2p^2+\Delta_x)^{-2}. \tag{20a}$$

Integrals with three factors can be reduced to those involving two by using (14a) again. They, therefore, lead to integrals with two parameters (e.g., see application to radiative correction to scattering below).

The methods of calculation given in this paper are deceptively simple when applied to the lower order processes. For processes of increasingly higher orders the complexity and difficulty increases rapidly, and these methods soon become impractical in their present form.

A. Self-Energy

The self-energy integral (19) is

$$(e^2/\pi i)\int\gamma_\mu(p-k-m)^{-1}\gamma_\mu k^{-2}d^4kC(k^2), \tag{19}$$

so that it requires that we find (using the principle of (8a)) the integral on L from 0 to λ^2 of

$$\int\gamma_\mu(p-k+m)\gamma_\mu d^4k(k^2-L)^{-2}(k^2-2p\cdot k)^{-1},$$

since $(p-k)^2-m^2=k^2-2p\cdot k$, as $p^2=m^2$. This is of the form (16a) with $\Delta_1=L$, $p_1=0$, $\Delta_2=0$, $p_2=p$ so that (18a) gives, since $p_x=(1-x)p$, $\Delta_x=xL$,

$$(8i)\int(1;k_\sigma)d^4k(k^2-L)^{-2}(k^2-2p\cdot k)^{-1}=\int_0^1(1;(1-x)p_\sigma)2xdx((1-x)^2m^2+xL)^{-1},$$

or performing the integral on L, as in (8),

$$(8i)\int(1;k_\sigma)d^4kk^{-2}C(k^2)(k^2-2p\cdot k)^{-1}=\int_0^1(1;(1-x)p_\sigma)2dx\ln\frac{x\lambda^2+(1-x)^2m^2}{(1-x)^2m^2}.$$

Assuming now that $\lambda^2\gg m^2$ we neglect $(1-x)^2m^2$ relative to $x\lambda^2$ in the argument of the logarithm, which then becomes $(\lambda^2/m^2)(x/(1-x)^2)$. Then since $\int_0^1dx\ln(x(1-x)^{-2})=1$ and $\int_0^1(1-x)dx\ln(x(1-x)^{-2})=-(1/4)$ find

$$(8i)\int(1;k_\sigma)k^{-2}C(k^2)d^4k(k^2-2p\cdot k)^{-1}=\left(2\ln\frac{\lambda^2}{m^2}+2;\ p_\sigma\left(\ln\frac{\lambda^2}{m^2}-\frac{1}{2}\right)\right),$$

so that substitution into (19) (after the $(p-k-m)^{-1}$ in (19) is replaced by $(p-k+m)(k^2-2p\cdot k)^{-1}$) gives

$$(19)=(e^2/8\pi)\gamma_\mu[(p+m)(2\ln(\lambda^2/m^2)+2)-p(\ln(\lambda^2/m^2)-\tfrac{1}{2})]\gamma_\mu$$
$$=(e^2/8\pi)[8m(\ln(\lambda^2/m^2)+1)-p(2\ln(\lambda^2/m^2)+5)], \tag{20}$$

using (4a) to remove the γ_μ's. This agrees with Eq. (20) of the text, and gives the self-energy (21) when p is replaced by m.

B. Corrections to Scattering

The term (12) in the radiationless scattering, after rationalizing the matrix denominators and using $p_1^2=p_2^2=m^2$ requires the integrals (9a), as we have discussed. This is an integral with three denominators which we do in two stages. First the factors $(k^2-2p_1\cdot k)$ and $(k^2-2p_2\cdot k)$ are combined by a parameter y;

$$(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=\int_0^1 dy(k^2-2p_y\cdot k)^{-2},$$

from (14a) where

$$p_y=yp_1+(1-y)p_2. \tag{21a}$$

We therefore need the integrals

$$(8i)\int(1;k_\sigma;k_\sigma k_\tau)d^4k(k^2-L)^{-2}(k^2-2p_y\cdot k)^{-2}, \tag{22a}$$

which we will then integrate with respect to y from 0 to 1. Next we do the integrals (22a) immediately from (20a) with $p=p_y$, $\Delta=0$:

$$(22a)=-\int_0^1\int_0^1(1;xp_{y\sigma};x^2p_{y\sigma}p_{y\tau}-\tfrac{1}{2}\delta_{\sigma\tau}(x^2p_y^2+(1-x)L))2x(1-x)dx(x^2p_y^2+L(1-x))^{-2}dy.$$

We now turn to the integrals on L as required in (8a). The first term, (1), in $(1;k_\sigma;k_\sigma k_\tau)$ gives no trouble for large L, but if L is put equal to zero there results $x^{-2}p_y^{-2}$ which leads to a diverging integral on x as $x\to 0$. This infra-red catastrophe is analyzed by using $\lambda_{\min}^2$ for the lower limit of the L integral. For the last term the upper limit of L must be kept as λ^2. Assuming $\lambda_{\min}^2\ll p_y^2\ll\lambda^2$ the x integrals which remain are trivial, as in the self-energy case. One finds

$$-(8i)\int(k^2-\lambda_{\min}^2)^{-1}d^4kC(k^2-\lambda_{\min}^2)(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=\int_0^1 p_y^{-2}dy\ln(p_y^2/\lambda_{\min}^2) \tag{23a}$$

$$-(8i)\int k_\sigma k^{-2}d^4kC(k^2)(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=2\int_0^1 p_{y\sigma}p_y^{-2}dy, \tag{24a}$$

$$-(8i)\int k_\sigma k_\tau k^{-2}d^4kC(k^2)(k^2-2p_1\cdot k)^{-1}(k^2-2p_2\cdot k)^{-1}=\int_0^1 p_{y\sigma}p_{y\tau}p_y^{-2}dy-\tfrac{1}{2}\delta_{\sigma\tau}\int_0^1 dy\ln(\lambda^2p_y^{-2})+\tfrac{1}{4}\delta_{\sigma\tau}. \tag{25a}$$

The integrals on y give,

$$\int_0^1 p_y^{-2}dy\ln(p_y^2\lambda_{\min}^{-2})=4(m^2\sin2\theta)^{-1}\left[\theta\ln(m\lambda_{\min}^{-1})-\int_0^\theta\alpha\tan\alpha d\alpha\right], \tag{26a}$$

$$\int_0^1 p_{y\sigma}p_y^{-2}dy=\theta(m^2\sin2\theta)^{-1}(p_{1\sigma}+p_{2\sigma}), \tag{27a}$$

$$\int_0^1 p_{y\sigma}p_{y\tau}p_y^{-2}dy=\theta(2m^2\sin2\theta)^{-1}(p_{1\sigma}+p_{1\tau})(p_{2\sigma}+p_{2\tau})+q^{-2}q_\sigma q_\tau(1-\theta\operatorname{ctn}\theta), \tag{28a}$$

$$\int_0^1 dy\ln(\lambda^2p_y^{-2})=\ln(\lambda^2/m^2)+2(1-\theta\operatorname{ctn}\theta). \tag{29a}$$

These integrals on y were performed as follows. Since $\not p_2=\not p_1+\not q$ where q is the momentum carried by the potential, it follows from $p_2^2=p_1^2=m^2$ that $2p_1\cdot q=-q^2$ so that since $\not p_y=\not p_1+\not q(1-y)$, $p_y^2=m^2-q^2y(1-y)$. The substitution $2y-1=\tan\alpha/\tan\theta$ where θ is defined by $4m^2\sin^2\theta=q^2$ is useful for it means $p_y^2=m^2\sec^2\alpha/\sec^2\theta$ and $p_y^{-2}dy=(m^2\sin 2\theta)^{-1}d\alpha$ where α goes from $-\theta$ to $+\theta$.

These results are substituted into the original scattering formula (2a), giving (22). It has been simplified by frequent use of the fact that $\not p_1$ operating on the initial state is m, and likewise $\not p_2$ when it appears at the left is replacable by m. (Thus, to simplify:

$$\gamma_\mu\not p_2 a\not p_1\gamma_\mu=-2\not p_1 a\not p_2 \text{ by (4a)},$$
$$=-2(\not p_2-\not q)a(\not p_1+\not q)=-2(m-\not q)a(m+\not q).$$

A term like $\not q a\not q=-q^2a+2(a\cdot q)\not q$ is equivalent to just $-q^2a$ since $\not q=\not p_2-\not p_1=m-m$ has zero matrix element.) The renormalization term requires the corresponding integrals for the special case $q=0$.

C. Vacuum Polarization

The expressions (32) and (32′) for $J_{\mu\nu}$ in the vacuum polarization problem require the calculation of the integral

$$J_{\mu\nu}(m^2)=-\frac{e^2}{\pi i}\int Sp[\gamma_\mu(\not p-\tfrac{1}{2}\not q+m)\gamma_\nu(\not p+\tfrac{1}{2}\not q+m)]d^4p$$
$$\times((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}, \quad (32)$$

where we have replaced p by $p-\frac{1}{2}q$ to simplify the calculation somewhat. We shall indicate the method of calculation by studying the integral,

$$I(m^2)=\int p_\sigma p_\tau d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}.$$

The factors in the denominator, $p^2-p\cdot q-m^2+\frac{1}{4}q^2$ and $p^2+p\cdot q-m^2+\frac{1}{4}q^2$ are combined as usual by (8a) but for symmetry we substitute $x=\frac{1}{2}(1+\eta)$, $(1-x)=\frac{1}{2}(1-\eta)$ and integrate η from -1 to $+1$:

$$I(m^2)=\int_{-1}^{+1}p_\sigma p_\tau d^4p(p^2-\eta p\cdot q-m^2+\tfrac{1}{4}q^2)^{-2}d\eta/2. \quad (30a)$$

But the integral on p will not be found in our list for it is badly divergent. However, as discussed in Section 7, Eq. (32′) we do not wish $I(m^2)$ but rather $\int_0^\infty[I(m^2)-I(m^2+\lambda^2)]G(\lambda)d\lambda$. We can calculate the difference $I(m^2)-I(m^2+\lambda^2)$ by first calculating the derivative $I'(m^2+L)$ of I with respect to m^2 at m^2+L and later integrating L from zero to λ^2. By differentiating (30a), with respect to m^2 find,

$$I'(m^2+L)=\int_{-1}^{+1}p_\sigma p_\tau d^4p(p^2-\eta p\cdot q-m^2-L+\tfrac{1}{4}q^2)^{-3}d\eta.$$

This still diverges, but we can differentiate again to get

$$I''(m^2+L)=3\int_{-1}^{+1}p_\sigma p_\tau d^4p(p^2-\eta p\cdot q-m^2-L+\tfrac{1}{4}q^2)^{-4}d\eta$$
$$=-(8i)^{-1}\int_{-1}^{+1}(\tfrac{1}{4}\eta^2q_\sigma q_\tau D^{-2}-\tfrac{1}{2}\delta_{\sigma\tau}D^{-1})d\eta \quad (31a)$$

(where $D=\frac{1}{4}(\eta^2-1)q^2+m^2+L$), which now converges and has been evaluated by (13a) with $p=\frac{1}{2}\eta q$ and $\Delta=m^2+L-\frac{1}{4}q^2$. Now to get I' we may integrate I'' with respect to L as an indefinite integral and *we may choose any convenient arbitrary constant.* This is because a constant C in I' will mean a term $-C\lambda^2$ in $I(m^2)-I(m^2+\lambda^2)$ which vanishes since we will integrate the results times $G(\lambda)d\lambda$ and $\int_0^\infty\lambda^2G(\lambda)d\lambda=0$. This means that the logarithm appearing on integrating L in (31a) presents no problem. We may take

$$I'(m^2+L)=(8i)^{-1}\int_{-1}^{+1}[\tfrac{1}{4}\eta^2q_\sigma q_\tau D^{-1}+\tfrac{1}{2}\delta_{\sigma\tau}\ln D]d\eta+C\delta_{\sigma\tau},$$

a subsequent integral on L and finally on η presents no new problems. There results

$$-(8i)\int p_\sigma p_\tau d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}$$
$$=(q_\sigma q_\tau-\delta_{\sigma\tau}q^2)\left[\frac{1}{9}-\frac{4m^2-q^2}{3q^2}\left(1-\frac{\theta}{\tan\theta}\right)+\tfrac{1}{6}\ln\frac{\lambda^2}{m^2}\right]$$
$$+\delta_{\sigma\tau}[(\lambda^2+m^2)\ln(\lambda^2m^{-2}+1)-C'\lambda^2], \quad (32a)$$

where we assume $\lambda^2\gg m^2$ and have put some terms into the arbitrary constant C' which is independent of λ^2 (but in principle could depend on q^2) and which drops out in the integral on $G(\lambda)d\lambda$. We have set $q^2=4m^2\sin^2\theta$.

In a very similar way the integral with m^2 in the numerator can be worked out. It is, of course, necessary to differentiate this m^2 also when calculating I' and I''. There results

$$-(8i)\int m^2d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}$$
$$=4m^2(1-\theta\operatorname{ctn}\theta)-q^2/3+2(\lambda^2+m^2)\ln(\lambda^2m^{-2}+1)-C''\lambda^2), \quad (33a)$$

with another unimportant constant C''. The complete problem requires the further integral,

$$-(8i)\int(1;p_\sigma)d^4p((p-\tfrac{1}{2}q)^2-m^2)^{-1}((p+\tfrac{1}{2}q)^2-m^2)^{-1}$$
$$=(1,0)(4(1-\theta\operatorname{ctn}\theta)+2\ln(\lambda^2m^{-2})). \quad (34a)$$

The value of the integral (34a) times m^2 differs from (33a), of course, because the results on the right are not actually the integrals on the left, but rather equal their actual value minus their value for $m^2=m^2+\lambda^2$.

Combining these quantities, as required by (32), dropping the constants C', C'' and evaluating the spur gives (33). The spurs are evaluated in the usual way, noting that the spur of any odd number of γ matrices vanishes and $Sp(AB)=Sp(BA)$ for arbitrary A, B. The $Sp(1)=4$ and we also have

$$\tfrac{1}{4}Sp[(\not p_1+m_1)(\not p_2-m_2)]=p_1\cdot p_2-m_1m_2, \quad (35a)$$

$$\tfrac{1}{4}Sp[(\not p_1+m_1)(\not p_2-m_2)(\not p_3+m_3)(\not p_4-m_4)]$$
$$=(p_1\cdot p_2-m_1m_2)(p_3\cdot p_4-m_3m_4)$$
$$-(p_1\cdot p_3-m_1m_3)(p_2\cdot p_4-m_2m_4)$$
$$+(p_1\cdot p_4-m_1m_4)(p_2\cdot p_3-m_2m_3), \quad (36a)$$

where $\not p_i$, m_i are arbitrary four-vectors and constants.

It is interesting that the terms of order $\lambda^2\ln\lambda^2$ go out, so that the charge renormalization depends only logarithmically on λ^2. This is not true for some of the meson theories. Electrodynamics is suspiciously unique in the mildness of its divergence.

D. More Complex Problems

Matrix elements for complex problems can be set up in a manner analogous to that used for the simpler cases. We give three illustrations; higher order corrections to the Møller scatter-

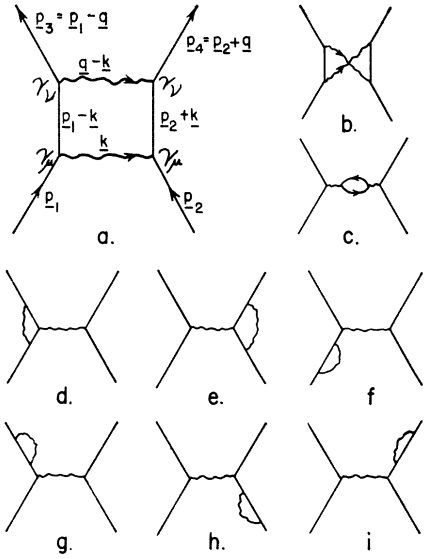

FIG. 8. The interaction between two electrons to order $(e^2/\hbar c)^2$. One adds the contribution of every figure involving two virtual quanta, Appendix D.

ing, to the Compton scattering, and the interaction of a neutron with an electromagnetic field.

For the Møller scattering, consider two electrons, one in state u_1 of momentum p_1 and the other in state u_2 of momentum p_2. Later they are found in states u_3, p_3 and u_4, p_4. This may happen (first order in $e^2/\hbar c$) because they exchange a quantum of momentum $q=p_1-p_3=p_4-p_2$ in the manner of Eq. (4) and Fig. 1. The matrix element for this process is proportional to (translating (4) to momentum space)

$$(\bar{u}_4\gamma_\mu u_2)(\bar{u}_3\gamma_\mu u_1)q^{-2}. \tag{37a}$$

We shall discuss corrections to (37a) to the next order in $e^2/\hbar c$. (There is also the possibility that it is the electron at 2 which finally arrives at 3, the electron at 1 going to 4 through the exchange of quantum of momentum p_3-p_2. The amplitude for this process, $(\bar{u}_4\gamma_\mu u_1)(\bar{u}_3\gamma_\mu u_2)(p_3-p_2)^{-2}$, must be subtracted from (37a) in accordance with the exclusion principle. A similar situation exists to each order so that we need consider in detail only the corrections to (37a), reserving to the last the subtraction of the same terms with 3, 4 exchanged.)

One reason that (37a) is modified is that two quanta may be exchanged, in the manner of Fig. 8a. The total matrix element for all exchanges of this type is

$$(e^2/\pi i)\int(\bar{u}_3\gamma_\nu(p_1-k-m)^{-1}\gamma_\mu u_1)(\bar{u}_4\gamma_\nu(p_2+k-m)^{-1}\gamma_\mu u_2)\cdot k^{-2}(q-k)^{-2}d^4k, \tag{38a}$$

as is clear from the figure and the general rule that electrons of momentum p contribute in amplitude $(p-m)^{-1}$ between interactions γ_μ, and that quanta of momentum k contribute k^{-2}. In integrating on d^4k and summing over μ and ν, we add all alternatives of the type of Fig. 8a. If the time of absorption, γ_μ, of the quantum k by electron 2 is later than the absorption, γ_ν, of $q-k$, this corresponds to the virtual state p_2+k being a positron (so that (38a) contains over thirty terms of the conventional method of analysis).

In integrating over all these alternatives we have considered all possible distortions of Fig. 8a which preserve the order of events along the trajectories. We have not included the possibilities corresponding to Fig. 8b, however. Their contribution is

$$(e^2/\pi i)\int(\bar{u}_3\gamma_\nu(p_1-k-m)^{-1}\gamma_\mu u_1)\times(\bar{u}_4\gamma_\mu(p_2+q-k-m)^{-1}\gamma_\nu u_2)k^{-2}(q-k)^{-2}d^4k, \tag{39a}$$

as is readily verified by labeling the diagram. The contributions of all possible ways that an event can occur are to be added. This means that one adds with equal weight the integrals corresponding to each topologically distinct figure.

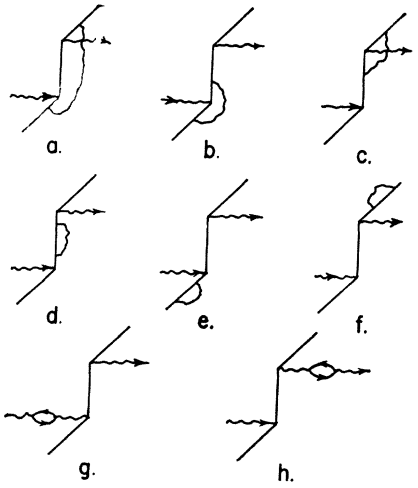

FIG. 9. Radiative correction to the Compton scattering term (a) of Fig. 5. Appendix D.

To this same order there are also the possibilities of Fig. 8d which give

$$(e^2/\pi i)\int(\bar{u}_3\gamma_\nu(p_3-k-m)^{-1}\gamma_\mu(p_1-k-m)^{-1}\gamma_\nu u_1)\times(\bar{u}_4\gamma_\mu u_2)k^{-2}q^{-2}d^4k.$$

This integral on k will be seen to be precisely the integral (12) for the radiative corrections to scattering, which we have worked out. The term may be combined with the renormalization terms resulting from the difference of the effects of mass change and the terms, Figs. 8f and 8g. Figures 8e, 8h, and 8i are similarly analyzed.

Finally the term Fig. 8c is clearly related to our vacuum polarization problem, and when integrated gives a term proportional to $(\bar{u}_4\gamma_\mu u_2)(\bar{u}_3\gamma_\nu u_1)J_{\mu\nu}q^{-4}$. If the charge is renormalized the term $\ln(\lambda/m)$ in $J_{\mu\nu}$ in (33) is omitted so there is no remaining dependence on the cut-off.

The only new integrals we require are the convergent integrals (38a) and (39a). They can be simplified by rationalizing the denominators and combining them by (14a). For example (38a) involves the factors $(k^2-2p_1\cdot k)^{-1}(k^2+2p_2\cdot k)^{-1}k^{-2}(q^2+k^2-2q\cdot k)^{-2}$. The first two may be combined by (14a) with a parameter x, and the second pair by an expression obtained by differentiation (15a) with respect to b and calling the parameter y. There results a factor $(k^2-2p_x\cdot k)^{-2}(k^2+yq^2-2yq\cdot k)^{-4}$ so that the integrals on d^4k now involve two factors and can be performed by the methods given earlier in the appendix. The subsequent integrals on the parameters x and y are complicated and have not been worked out in detail.

Working with charged mesons there is often a considerable reduction of the number of terms. For example, for the interaction between protons resulting from the exchange of two mesons only the term corresponding to Fig. 8b remains. Term 8a, for example, is impossible, for if the first proton emits a positive meson the second cannot absorb it directly for only neutrons can absorb positive mesons.

As a second example, consider the radiative correction to the Compton scattering. As seen from Eq. (15) and Fig. 5 this scattering is represented by two terms, so that we can consider the corrections to each one separately. Figure 9 shows the types of terms arising from corrections to the term of Fig. 5a. Calling k the momentum of the virtual quantum, Fig. 9a gives an integral

$$\int\gamma_\mu(p_2-k-m)^{-1}e_2(p_1+q_1-k-m)^{-1}e_1(p_1-k-m)^{-1}\gamma_\mu k^{-2}d^4k,$$

convergent without cut-off and reducible by the methods outlined in this appendix.

The other terms are relatively easy to evaluate. Terms b and c of Fig. 9 are closely related to radiative corrections (although somewhat more difficult to evaluate, for one of the states is not that of a free electron, $(p_1+q)^2\neq m^2$). Terms e, f are renormalization terms. From term d must be subtracted explicitly the effect of mass Δm, as analyzed in Eqs. (26) and (27) leading to (28) with $p'=p_1+q$, $a=e_2$, $b=e_1$. Terms g, h give zero since the vacuum polarization has zero effect on free light quanta, $q_1^2=0$, $q_2^2=0$. The total is insensitive to the cut-off λ.

The result shows an infra-red catastrophe, the largest part of the effect. When cut-off at $\lambda_{\min}$, the effect proportional to $\ln(m/\lambda_{\min})$ goes as

$$(e^2/\pi)\ln(m/\lambda_{\min})(1-2\theta\operatorname{ctn}2\theta), \tag{40a}$$

times the uncorrected amplitude, where $(p_2-p_1)^2=4m^2\sin^2\theta$. This is the same as for the radiative correction to scattering for a deflection p_2-p_1. This is physically clear since the long wave quanta are not effected by short-lived intermediate states. The infra-red effects arise[28] from a final adjustment of the field from the asymptotic coulomb field characteristic of the electron of

[28] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).

momentum $\boldsymbol{p}_1$ before the collision to that characteristic of an electron moving in a new direction $\boldsymbol{p}_2$ after the collision.

The complete expression for the correction is a very complicated expression involving transcendental integrals.

As a final example we consider the interaction of a neutron with an electromagnetic field in virtue of the fact that the neutron may emit a virtual negative meson. We choose the example of pseudoscalar mesons with pseudovector coupling. The change in amplitude due to an electromagnetic field $\boldsymbol{A}=\boldsymbol{a}\exp(-i\boldsymbol{q}\cdot x)$ determines the scattering of a neutron by such a field. In the limit of small $\boldsymbol{q}$ it will vary as $\boldsymbol{qa}-\boldsymbol{aq}$ which represents the interaction of a particle possessing a magnetic moment. The first-order interaction between an electron and a neutron is given by the same calculation by considering the exchange of a quantum between the electron and the nucleon. In this case a_μ is q^{-2} times the matrix element of γ_μ between the initial and final states of the electron, the states differing in momentum by $\boldsymbol{q}$.

The interaction may occur because the neutron of momentum $\boldsymbol{p}_1$ emits a negative meson becoming a proton which proton interacts with the field and then reabsorbs the meson (Fig. 10a). The matrix for this process is $(\boldsymbol{p}_2=\boldsymbol{p}_1+\boldsymbol{q})$,

$$\int(\gamma_5\boldsymbol{k})(\boldsymbol{p}_2-\boldsymbol{k}-M)^{-1}\boldsymbol{a}(\boldsymbol{p}_1-\boldsymbol{k}-M)^{-1}(\gamma_5\boldsymbol{k})(\boldsymbol{k}^2-\mu^2)^{-1}d^4k. \quad (41a)$$

Alternatively it may be the meson which interacts with the field. We assume that it does this in the manner of a scalar potential satisfying the Klein Gordon Eq. (35), (Fig. 10b)

$$-\int(\gamma_5\boldsymbol{k}_2)(\boldsymbol{p}_1-\boldsymbol{k}_1-M)^{-1}(\gamma_5\boldsymbol{k}_1)(\boldsymbol{k}_2^2-\mu^2)^{-1}\times(k_2\cdot a+k_1\cdot a)(\boldsymbol{k}_1^2-\mu^2)^{-1}d^4k_1, \quad (42a)$$

where we have put $\boldsymbol{k}_2=\boldsymbol{k}_1+\boldsymbol{q}$. The change in sign arises because the virtual meson is negative. Finally there are two terms arising from the $\gamma_5\boldsymbol{a}$ part of the pseudovector coupling (Figs. 10c, 10d)

$$\int(\gamma_5\boldsymbol{k})(\boldsymbol{p}_2-\boldsymbol{k}-M)^{-1}(\gamma_5\boldsymbol{a})(\boldsymbol{k}^2-\mu^2)^{-1}d^4k, \quad (43a)$$

and

$$\int(\gamma_5\boldsymbol{a})(\boldsymbol{p}_1-\boldsymbol{k}-M)^{-1}(\gamma_5\boldsymbol{k})(\boldsymbol{k}^2-\mu^2)^{-1}d^4k. \quad (44a)$$

Using convergence factors in the manner discussed in the section on meson theories each integral can be evaluated and the results combined. Expanded in powers of $\boldsymbol{q}$ the first term gives the magnetic moment of the neutron and is insensitive to the cut-off, the next gives the scattering amplitude of slow electrons on neutrons, and depends logarithmically on the cut-off.

The expressions may be simplified and combined somewhat before integration. This makes the integrals a little easier and also shows the relation to the case of pseudoscalar coupling. For example in (41a) the final $\gamma_5\boldsymbol{k}$ can be written as $\gamma_5(\boldsymbol{k}-\boldsymbol{p}_1+M)$ since $\boldsymbol{p}_1=M$ when operating on the initial neutron state. This is

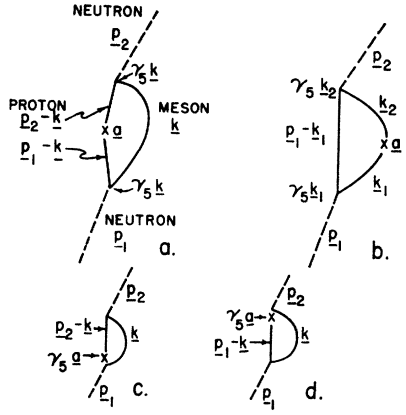

FIG. 10. According to the meson theory a neutron interacts with an electromagnetic potential $\boldsymbol{a}$ by first emitting a virtual charged meson. The figure illustrates the case for a pseudoscalar meson with pseudovector coupling. Appendix D.

$(\boldsymbol{p}_1-\boldsymbol{k}-M)\gamma_5+2M\gamma_5$ since γ_5 anticommutes with $\boldsymbol{p}_1$ and $\boldsymbol{k}$. The first term cancels the $(\boldsymbol{p}_1-\boldsymbol{k}-M)^{-1}$ and gives a term which just cancels (43a). In a like manner the leading factor $\gamma_5\boldsymbol{k}$ in (41a) is written as $-2M\gamma_5-\gamma_5(\boldsymbol{p}_2-\boldsymbol{k}-M)$, the second term leading to a simpler term containing no $(\boldsymbol{p}_2-k-M)^{-1}$ factor and combining with a similar one from (44a). One simplifies the $\gamma_5\boldsymbol{k}_1$ and $\gamma_5\boldsymbol{k}_2$ in (42a) in an analogous way. There finally results terms like (41a), (42a) but with pseudoscalar coupling $2M\gamma_5$ instead of $\gamma_5\boldsymbol{k}$, no terms like (43a) or (44a) and a remainder, representing the difference in effects of pseudovector and pseudoscalar coupling. The pseudoscalar terms do not depend sensitively on the cut-off, but the difference term depends on it logarithmically. The difference term affects the electron-neutron interaction but not the magnetic moment of the neutron.

Interaction of a proton with an electromagnetic potential can be similarly analyzed. There is an effect of virtual mesons on the electromagnetic properties of the proton even in the case that the mesons are neutral. It is analogous to the radiative corrections to the scattering of electrons due to virtual photons. The sum of the magnetic moments of neutron and proton for charged mesons is the same as the proton moment calculated for the corresponding neutral mesons. In fact it is readily seen by comparing diagrams, that for arbitrary $\boldsymbol{q}$, the scattering matrix to *first order in the electromagnetic potential* for a proton according to neutral meson theory is equal, if the mesons were charged, to the sum of the matrix for a neutron and the matrix for a proton. This is true, for any type or mixtures of meson coupling, to all orders in the coupling (neglecting the mass difference of neutron and proton).

THE DEVELOPMENT OF THE SPACE-TIME VIEW OF QUANTUM ELECTRODYNAMICS*

by

Richard P. Feynman

California Institute of Technology, Pasadena, California

Nobel Lecture, December 11, 1965.

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover all the tracks, to not worry about the blind alleys or to describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work, although, there has been in these days, some interest in this kind of thing. Since winning the prize is a personal thing, I thought I could be excused in this particular situation, if I were to talk personally about my relationship to quantum electrodynamics, rather than to discuss the subject itself in a refined and finished fashion. Furthermore, since there are three people who have won the prize in physics, if they are all going to be talking about quantum electrodynamics itself, one might become bored with the subject. So, what I would like to tell you about today are the sequence of events, really the sequence of ideas, which occurred, and by which I finally came out the other end with an unsolved problem for which I ultimately received a prize.

I realize that a truly scientific paper would be of greater value, but such a paper I could publish in regular journals. So, I shall use this Nobel Lecture as an opportunity to do something of less value, but which I cannot do elsewhere. I ask your indulgence in another manner. I shall include details of anecdotes which are of no value either scientifically, nor for understanding the development of ideas. They are included only to make the lecture more entertaining.

I worked on this problem about eight years until the final publication in 1947. The beginning of the thing was at the Massachusetts Institute of Technology, when I was an undergraduate student reading about the known physics, learning slowly about all these things that people were worrying about, and realizing ultimately that the fundamental problem of the day was that the quantum theory of electricity and magnetism was not completely satisfactory. This I gathered from books like those of Heitler and Dirac. I was inspired by the remarks in these books; not by the parts in which everything was proved and demonstrated carefully and calculated, because I couldn't understand those very well. At the young age what I could understand were

* This document is a revised version of Feynman's Lecture, with amendments made by Michael D. Godfrey and Michael A. Gottlieb (email: godfrey@isl.stanford.edu and codelieb@caltech.edu).

the remarks about the fact that this doesn't make any sense, and the last sentence of the book of Dirac I can still remember, "It seems that some essentially new physical ideas are here needed." So, I had this as a challenge and an inspiration. I also had a personal feeling, that since they didn't get a satisfactory answer to the problem I wanted to solve, I don't have to pay a lot of attention to what they did do.

I did gather from my readings, however, that two things were the source of the difficulties with the quantum electrodynamical theories. The first was an infinite energy of interaction of the electron with itself. And this difficulty existed even in the classical theory. The other difficulty came from some infinities which had to do with the infinite number of degrees of freedom in the field. As I understood it at the time (as nearly as I can remember) this was simply the difficulty that if you quantized the harmonic oscillators of the field (say in a box) each oscillator has a ground state energy of $\frac{1}{2}\hbar\omega$ and there is an infinite number of modes in a box of ever increasing frequency ω, and therefore there is an infinite energy in the box. I now realize that that wasn't a completely correct statement of the central problem; it can be removed simply by changing the zero from which energy is measured. At any rate, I believed that the difficulty arose somehow from a combination of the electron acting on itself and the infinite number of degrees of freedom of the field.

Well, it seemed to me quite evident that the idea that a particle acts on itself, that the electrical force acts on the same particle that generates it, is not a necessary one—it is sort of a silly one, as a matter of fact. And, so I suggested to myself, that electrons cannot act on themselves, they can only act on other electrons. That means there is no field at all. You see, if all charges contribute to making a single common field, and if that common field acts back on all the charges, then each charge must act back on itself. Well, that was where the mistake was, there was no field. It was just that when you shook one charge, another would shake later. There was a direct interaction between charges, albeit with a delay. The law of force connecting the motion of one charge with another would just involve a delay. Shake this one, that one shakes later. The sun atom shakes; my eye electron shakes eight minutes later, because of a direct interaction across.

Now, this has the attractive feature that it solves both problems at once. First, I can say immediately, I don't let the electron act on itself, I just let this act on that, hence, no self-energy! Secondly, there is not an infinite number of degrees of freedom in the field. There is no field at all; or if you insist on thinking in terms of ideas like that of a field, this field is always completely determined by the action of the particles which produce it. You shake this particle, it shakes that one, but if you want to think in a field way, the field, if it's there, would be entirely determined by the matter which generates it, and therefore, the field does not have any *independent* degrees of freedom and the infinities from the degrees of freedom would then be removed. As a matter of fact, when we look out anywhere and see light, we can always "see" some matter as the source of the light. We don't just see light (except recently some radio reception has been found with no apparent material source).

You see then that my general plan was to first solve the classical problem, to get rid of the infinite self-energies in the classical theory, and to hope that when I made a quantum theory of it, everything would just be fine.

That was the beginning, and the idea seemed so obvious to me and so elegant that I fell deeply in love with it. And, like falling in love with a woman, it is only possible if you do not know much about her, so you cannot see her faults. The faults will become apparent later, but after the love is strong enough to hold you to her. So, I was held to this theory, in spite of all difficulties, by my youthful enthusiasm.

Then I went to graduate school and somewhere along the line I learned what was wrong with the idea that an electron does not act on itself. When you accelerate an electron it radiates energy and you have to do extra work to account for that energy. The extra force against which this work is done is called the force of radiation resistance. The origin of this extra force was identified in those days, following Lorentz, as the action of the electron itself. The first term of this action, of the electron on itself, gave a kind of inertia (not quite relativistically satisfactory). But that inertia-like term was infinite for a point-charge. Yet the next term in the sequence gave an energy loss rate, which for a point-charge agrees exactly with the rate you get by calculating how much energy is radiated. So, the force of radiation resistance, which is absolutely necessary for the conservation of energy, would disappear if I said that a charge could not act on itself.

So, I learned in the interim when I went to graduate school the glaringly obvious fault of my own theory. But, I was still in love with the original theory, and was still thinking that with it lay the solution to the difficulties of quantum electrodynamics. So, I continued to try on and off to save it somehow. I must have some action develop on a given electron when I accelerate it to account for radiation resistance. But, if I let electrons only act on other electrons the only possible source for this action is another electron in the world. So, one day, when I was working for Professor Wheeler and could no longer solve the problem that he had given me, I thought about this again and I calculated the following: Suppose I have two charges—I shake the first charge, which I think of as a source and this makes the second one shake, but the second one shaking produces an effect back on the source. And so, I calculated how much that effect back on the first charge was, hoping it might add up to the force of radiation resistance. It didn't come out right, of course, but I went to Professor Wheeler and told him my ideas. He said,—yes, but the answer you get for the problem with the two charges that you just mentioned will, unfortunately, depend upon the charge and the mass of the second charge and will vary inversely as the square of the distance R, between the charges, while the force of radiation resistance depends on none of these things. I thought, surely, he had computed it himself, but now having become a professor, I know that one can be wise enough to see immediately what some graduate student takes several weeks to develop. He also pointed out something else that bothered me, that if we had a situation with many charges all around the original source at roughly uniform density and if we added the effect of all the surrounding charges the inverse R^2 would be compensated by the R^2 in the volume element and we would get a result proportional to the thickness of the layer, which would go to infinity. That is, one would have an infinite total effect back at the source. And, finally he said to me, and you forgot something else, when you accelerate the first charge, the second acts later, and then the reaction back here at the source would be still later. In other words, the action occurs at the wrong time. I

suddenly realized what a stupid fellow I am, for what I had described and calculated was just ordinary reflected light, not radiation reaction.

But, as I was stupid, so was Professor Wheeler that much more clever. For he then went on to give a lecture as though he had worked this all out before and was completely prepared, but he had not, he worked it out as he went along. First, he said, let us suppose that the return action by the charges in the absorber reaches the source by advanced waves as well as by the ordinary retarded waves of reflected light; so that the law of interaction acts backward in time, as well as forward in time. I was enough of a physicist at that time not to say, "Oh, no, how could that be?" For today all physicists know from studying Einstein and Bohr, that sometimes an idea which looks completely paradoxical at first, if analyzed to completion in all detail and in experimental situations, may, in fact, not be paradoxical. So, it did not bother me any more than it bothered Professor Wheeler to use advance waves for the back reaction—a solution of Maxwell's equations, which previously had not been physically used.

Professor Wheeler used advanced waves to get the reaction back at the right time and then he suggested this: If there were lots of electrons in the absorber, there would be an index of refraction n, so, the retarded waves coming from the source would have their wavelengths slightly modified in going through the absorber. Now, if we shall assume that the advanced waves come back from the absorber without an index—why? I don't know, let's assume they come back without an index—then, there will be a gradual shifting in phase between the return and the original signal so that we would only have to figure that the contributions act as if they come from only a finite thickness, that of the first wave zone. (More specifically, up to that depth where the phase in the medium is shifted appreciably from what it would be in vacuum, a thickness proportional to $\lambda/(n-1)$.) Now, the less the number of electrons in here, the less each contributes, but the thicker will be the layer that effectively contributes because with less electrons, the index differs less from 1. The higher the charges of these electrons, the more each contributes, but the thinner the effective layer, because the index would be higher. And when we estimated it, (calculated without being careful to keep the correct numerical factor) sure enough, it came out that the action back at the source was completely independent of the properties of the charges that were in the surrounding absorber. Further, it was of just the right character to represent radiation resistance, but we were unable to see if it was just exactly the right size. He sent me home with orders to figure out exactly how much advanced and how much retarded wave we need to get the thing to come out numerically right, and after that, figure out what happens to the advanced effects that you would expect if you put a test charge here close to the source? For if all charges generate advanced, as well as retarded effects, why would that test charge not be affected by the advanced waves from the source?

I found that you get the right answer if you use half-advanced and half-retarded as the field generated by each charge. That is, one is to use the solution of Maxwell's equation which is symmetrical in time and that the reason we got no advanced effects at a point close to the source in spite of the fact that the source was producing an advanced field is this: Suppose the source is surrounded by a spherical absorbing wall

ten light seconds away, and that the test charge is one second to the right of the source. Then the source is as much as eleven seconds away from some parts of the wall and only nine seconds away from other parts. The source acting at time $t = 0$ induces motions in the wall at time $t = +10$. Advanced effects from this can act on the test charge as early as eleven seconds earlier, or at $t = -1$. This is just at the time that the direct advanced waves from the source should reach the test charge, and it turns out the two effects are exactly equal and opposite and cancel out! At the later time $t = +1$ effects on the test charge from the source and from the walls are again equal, but this time are of the same sign and add to convert the half-retarded wave of the source to full retarded strength.

Thus, it became clear that there was the possibility that if we assume all actions are via half-advanced and half-retarded solutions of Maxwell's equations and that all sources are surrounded by material absorbing all the the light which is emitted, then we could account for radiation resistance as a direct action of the charges of the absorber acting back by advanced waves on the source.

Many months were devoted to checking all these points. I worked to show that everything is independent of the shape of the container, and so on, that the laws are exactly right, and that the advanced effects really cancel in every case. We always tried to increase the efficiency of our demonstrations, and to see with more and more clarity why it works. I won't bore you by going through the details of this. Because of our using advanced waves we also had many apparent paradoxes, which we gradually reduced one by one, and saw that there was in fact no logical difficulty with the theory. It was perfectly satisfactory.

We also found that we could reformulate this thing in another way, and that is by a principle of least action. Since my original plan was to describe everything directly in terms of particle motions, it was my desire to represent this new theory without saying anything about fields. It turned out that we found a form for an action directly involving the motions of the charges only, which upon variation would give the equations of motion of these charges. The expression for this action A is

$$A = \sum_i m_i \int \left(\dot{X}^i_\mu \dot{X}^i_\mu\right)^{\frac{1}{2}} d\alpha_i + \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} e_i e_j \int\int \delta(I^2_{ij}) \dot{X}^i_\mu(\alpha_i) \dot{X}^j_\mu(\alpha_j) d\alpha_i d\alpha_j \qquad (1)$$

where

$$I^2_{ij} = [X^i_\mu(\alpha_i) - X^j_\mu(\alpha_j)][X^i_\mu(\alpha_i) - X^j_\mu(\alpha_j)]$$

and where $X^i_\mu(\alpha_i)$ is the four-vector position of the i^{th} particle as a function of some parameter α_i, and $\dot{X}^i_\mu(\alpha_i)$ is $dX^i_\mu(\alpha_i)/d\alpha_i$. The first term is the integral of proper time, the ordinary action of relativistic mechanics of free particles of mass m_i. (We sum in the usual way on the repeated index μ.) The second term represents the electrical interaction of the charges. It is summed over each pair of charges (the factor $\frac{1}{2}$ is to count each pair once, the term $i = j$ is omitted to avoid self-action). The interaction is a double integral over a δ-function of the square of the space-time interval I^2 between two points on the paths. Thus, interaction occurs only when this interval vanishes, that is, along light cones.

The fact that the interaction is exactly one-half advanced and one-half retarded meant that we could write such a principle of least action, whereas interaction via retarded waves alone cannot be written in such a way.

So, all of classical electrodynamics was contained in this very simple form. It looked good, and therefore, it was undoubtedly true, at least to the beginner. It automatically gave half-advanced and half-retarded effects and it was without fields. By omitting the term in the sum when $i = j$, I omitted self-interaction and no longer had any infinite self-energy. This then was the hoped-for solution to the problem of ridding classical electrodynamics of the infinities.

It turns out, of course, that you can reinstate fields if you wish to, but you have to keep track of the field produced by each particle separately. This is because to find the right field to act on a given particle, you must exclude the field that it creates itself. A single universal field to which all contribute will not do. This idea had been suggested earlier by Frenkel and so we called these Frenkel fields. This theory which allowed only particles to act on each other was equivalent to Frenkel's fields using half-advanced and half-retarded solutions.

There were several suggestions for interesting modifications of electrodynamics. We discussed lots of them, but I shall report on only one. It was to replace this δ-function in the interaction by another function, say, $f(I_{ij}^2)$, which is not infinitely sharp. Instead of having the action occur only when the interval between the two charges is exactly zero, we would replace the δ-function of I^2 by a narrow peaked thing. Let's say that $f(Z)$ is large only near $Z = 0$ and has width of order a^2. Interactions will now occur when $T^2 - R^2$ is roughly of order a^2, where T is the time difference and R is the separation of the charges. This might look like it disagrees with experience, but if a is some small distance, like 10^{-13}cm, it says that the time delay T in action is roughly $\sqrt{R^2 \pm a^2}$ or approximately, if R is much larger than a, $T = R \pm a^2/2R$. This means that the deviation of time T from the ideal theoretical time R of Maxwell, gets smaller and smaller, the further the pieces are apart. Therefore, all theories involved in analyzing generators, motors, etc., in fact, all of the tests of electrodynamics that have been available since Maxwell's time, would be adequately satisfied if a were 10^{-13}cm. If R is of the order of a centimeter this deviation in T is only 10^{-26} seconds. So, it was possible, also, to change the theory in a simple manner and to still agree with all observations of classical electrodynamics. You have no clue of precisely what function to put in for f, but it was an interesting possibility to keep in mind when developing quantum electrodynamics.

It also occurred to us that if we did that (replace δ by f) we could reinstate the term $i = j$ in the sum because this would now represent, in a relativistically invariant fashion, a finite action of a charge on itself. In fact, it was possible to prove that if we did do such a thing, the main effect of the self-action (for not too rapid accelerations) would be to produce a modification of the mass. In fact, there need be no mass m_i term; all the mechanical mass could be electromagnetic self-action. So, if you would like, we could also have another theory with a still simpler expression for the action A. In expression (1) only the second term is kept, the sum extended over all i and j, and some function f replaces δ. Such a simple form could represent all of classical electrodynamics, which aside from gravitation is essentially all of classical physics.

Although it may sound confusing, I am describing several different alternative theories at once. The important thing to note is that at this time we had all these in mind as different possibilities. There were several possible solutions of the difficulty of classical electrodynamics, any one of which might serve as a good starting point to the solution of the difficulties of quantum electrodynamics.

I would also like to emphasize that by this time I was becoming used to a physical point of view different from the more customary point of view. In the customary view, things are discussed as a function of time in very great detail. For example, you have the field at this moment, a differential equation gives you the field at the next moment and so on; a method, which I shall call the Hamiltonian method, the time differential method. We have, instead (in (1) say) a thing that describes the character of the path throughout all of space and time. The behavior of nature is determined by saying her whole space-time path has a certain character. For an action like (1) the equations obtained by variation of $X^i_\mu(\alpha_i)$ are no longer at all easy to get back into Hamiltonian form. If you wish to use as variables only the coordinates of particles, then you can talk about the property of the paths—but the path of one particle at a given time is affected by the path of another at a different time. If you try to describe, therefore, things differentially, telling what the present conditions of the particles are, and how these present conditions will affect the future, you see, it is impossible with particles alone, because something the particle did in the past is going to affect the future.

Therefore, you need a lot of bookkeeping variables to keep track of what the particle did in the past. These are called field variables. You will, also, have to tell what the field is at this present moment, if you are to be able to see later what is going to happen. From the overall space-time view of the least action principle, the field disappears as nothing but bookkeeping variables insisted on by the Hamiltonian method.

As a by-product of this same view, I received a telephone call one day at the graduate college at Princeton from Professor Wheeler, in which he said, "Feynman, I know why all electrons have the same charge and the same mass." "Why?" "Because, they are all the same electron!" And, then he explained on the telephone, "suppose that the world lines which we were ordinarily considering before in time and space—instead of only going up in time were a tremendous knot, and then, when we cut through the knot, by the plane corresponding to a fixed time, we would see many, many world lines and that would represent many electrons, except for one thing. If in one section this is an ordinary electron world line, in the section in which it reversed itself and is coming back from the future we have the wrong sign to the proper time—to the proper four velocities—and that's equivalent to changing the sign of the charge, and, therefore, that part of a path would act like a positron." "But, Professor," I said, "there aren't as many positrons as electrons." "Well, maybe they are hidden in the protons or something," he said. I did not take the idea that all the electrons were the same one from him as seriously as I took the observation that positrons could simply be represented as electrons going from the future to the past in a back section of their world lines. That, I stole!

To summarize, when I was done with this, as a physicist I had gained two things. One, I knew many different ways of formulating classical electrodynamics, with many

different mathematical forms. I got to know how to express the subject every which way. Second, I had a point of view—the overall space-time point of view—and a disrespect for the Hamiltonian method of describing physics.

I would like to interrupt here to make a remark. The fact that electrodynamics can be written in so many ways—the differential equations of Maxwell, various minimum principles with fields, minimum principles without fields, all different kinds of ways, was something I knew, but I have never understood. It always seems odd to me that the fundamental laws of physics, when discovered, can appear in so many different forms that are not apparently identical at first, but, with a little mathematical fiddling you can show the relationship. An example of that is the Schrödinger equation and the Heisenberg formulation of quantum mechanics. I don't know why this is—it remains a mystery, but it was something I learned from experience. There is always another way to say the same thing that doesn't look at all like the way you said it before. I don't know what the reason for this is. I think it is somehow a representation of the simplicity of nature. A thing like the inverse square law is just right to be represented by the solution of Poisson's equation, which, therefore, is a very different way to say the same thing that doesn't look at all like the way you said it before. I don't know what it means, that nature chooses these curious forms, but maybe that is a way of defining simplicity. Perhaps a thing is simple if you can describe it fully in several different ways without immediately knowing that you are describing the same thing.

I was now convinced that since we had solved the problem of classical electrodynamics (and completely in accordance with my program from M.I.T., only direct interaction between particles, in a way that made fields unnecessary) that everything was definitely going to be all right. I was convinced that all I had to do was make a quantum theory analogous to the classical one and everything would be solved.

So, the problem is only to make a quantum theory which has as its classical analog this expression (1). Now, there is no unique way to make a quantum theory from classical mechanics, although all the textbooks make believe there is. What they would tell you to do, was find the momentum variables and replace them by $(\hbar/i)(\partial/\partial x)$: but I couldn't find a momentum variable, as there wasn't any.

The character of quantum mechanics of the day was to write things in the famous Hamiltonian way—in the form of a differential equation, which described how the wave function changes from instant to instant, and in terms of an operator, H. If the classical physics could be reduced to a Hamiltonian form, everything was all right. Now, least action does not imply a Hamiltonian form if the action is a function of anything more than positions and velocities at the same moment. If the action is of the form of the integral of a function, (usually called the Lagrangian) of the velocities and positions at the same time

$$S = \int L(\dot{x}, x)dt \tag{2}$$

then you can start with the Lagrangian and then create a Hamiltonian and work out the quantum mechanics, more or less uniquely. But this thing (1) involves the key variables, positions, at different times and therefore, it was not obvious what to do to make the quantum-mechanical analog.

I tried—I would struggle in various ways. One of them was this: if I had harmonic oscillators interacting with a delay in time, I could work out what the normal modes were and guess that the quantum theory of the normal modes was the same as for simple oscillators and kind of work my way back in terms of the original variables. I succeeded in doing that, and I hoped then to generalize to other than a harmonic oscillator, but I learned to my regret something, which many people have learned. The harmonic oscillator is too simple; very often you can work out what it should do in quantum theory without getting much of a clue as to how to generalize your results to other systems.

So that didn't help me very much, but when I was struggling with this problem, I went to a beer party in the Nassau Tavern in Princeton. There was a gentleman, newly arrived from Europe (Herbert Jehle) who came and sat next to me. Europeans are much more serious than we are in America because they think that a good place to discuss intellectual matters is a beer party. So, he sat by me and asked, "what are you doing" and so on, and I said, "I'm drinking beer." Then I realized that he wanted to know what work I was doing and I told him I was struggling with this problem, and I simply turned to him and said, "listen, do you know any way of doing quantum mechanics, starting with action—where the action integral comes into the quantum mechanics?" "No," he said, "but Dirac has a paper in which the Lagrangian, at least, comes into quantum mechanics. I will show it to you tomorrow."

Next day we went to the Princeton Library—they have little rooms on the side to discuss things—and he showed me this paper. What Dirac said was the following: There is in quantum mechanics a very important quantity which carries the wave function from one time to another, besides the differential equation but equivalent to it, a kind of a kernel, which we might call $K(x', x)$, which carries the wave function $\psi(x)$ known at time t, to the wave function $\psi(x')$ at time, $t + \varepsilon$. Dirac points out that this function K was *analogous* to the quantity in classical mechanics that you would calculate if you took the exponential of $i\varepsilon$, multiplied by the Lagrangian $L(\dot{x}, x)$ imagining that these two positions x, x' corresponded to t and $t + \varepsilon$. In other words,

$$K(x', x) \quad \text{is analogous to} \quad e^{i\varepsilon L\left(\frac{x'-x}{\varepsilon}, x\right)/\hbar}.$$

Professor Jehle showed me this, I read it, he explained it to me, and I said, "what does he mean, they are analogous; what does that mean, *analogous*? What is the use of that?" He said, "you Americans! You always want to find a use for everything!" I said, that I thought that Dirac must mean that they were equal. "No," he explained, "he doesn't mean they are equal." "Well," I said, "let's see what happens if we make them equal."

So I simply put them equal, taking the simplest example where the Lagrangian is $\frac{1}{2}M\dot{x}^2 - V(x)$ but soon found I had to put a constant of proportionality A in, suitably adjusted. When I substituted $Ae^{i\varepsilon L/\hbar}$ for K to get

$$\psi(x', t + \varepsilon) = \int A\, exp\left[\frac{i\varepsilon}{\hbar} L\Big(\frac{x' - x}{\varepsilon}, x\Big)\right] \psi(x, t) dx \tag{3}$$

and just calculated things out by Taylor series expansion, out came the Schrödinger equation. So, I turned to Professor Jehle, not really understanding, and said, "well,

you see Professor Dirac meant that they were proportional." Professor Jehle's eyes were bugging out—he had taken out a little notebook and was rapidly copying it down from the blackboard, and said, "no, no, this is an important discovery. You Americans are always trying to find out how something can be used. That's a good way to discover things!" So, I thought I was finding out what Dirac meant, but, as a matter of fact, had made the discovery that what Dirac thought was analogous, was, in fact, equal. I had then, at least, the connection between the Lagrangian and quantum mechanics, but still with wave functions and infinitesimal times.

It must have been a day or so later, when I was lying in bed thinking about these things, that I imagined what would happen if I wanted to calculate the wave function at a finite interval later.

I would put one of these factors $e^{i\varepsilon L}$ in here, and that would give me the wave functions the next moment, $t+\varepsilon$, and then I could substitute that back into (3) to get another factor of $e^{i\varepsilon L}$ and give me the wave function the next moment, $t+2\varepsilon$, and so on and so on. In that way I found myself thinking of a large number of integrals, one after the other in sequence. In the integrand was the product of the exponentials, which, of course, was the exponential of the sum of terms like εL. Now, L is the Lagrangian and ε is like the time interval dt, so that if you took a sum of such terms, that's exactly like an integral. That's like Riemann's formula for the integral $\int L dt$; you just take the value at each point and add them together. We are to take the limit as $\varepsilon \to 0$, of course. Therefore, the connection between the wave function of one instant and the wave function of another instant a finite time later could be obtained by an infinite number of integrals, (because ε goes to zero, of course) of exponential $(iS/\hbar)$, where S is the action expression (2). At last, I had succeeded in representing quantum mechanics directly in terms of the action S.

This led later on to the idea of the amplitude for a path; that for each possible way that the particle can go from one point to another in space-time, there's an amplitude. That amplitude is e to the $(i/\hbar)$ times the action for the path. Amplitudes from various paths superpose by addition. This then is another, a third, way of describing quantum mechanics, which looks quite different than that of Schrödinger or Heisenberg, but which is equivalent to them.

Now immediately after making a few checks on this thing, what I wanted to do, of course, was to substitute the action (1) for the other (2). The first trouble was that I could not get the thing to work with the relativistic case of spin one-half. However, although I could deal with the matter only non-relativistically, I could deal with the light or the photon interactions perfectly well by just putting the interaction terms of (1) into any action, replacing the mass terms by the non-relativistic $(M\dot{x}^2/2)dt$. When the action has a delay, as it now had, and involved more than one time, I had to lose the idea of a wave function. That is, I could no longer describe the program as; given the amplitude for all positions at a certain time, compute the amplitude at another time. However, that didn't cause very much trouble. It just meant developing a new idea. Instead of wave functions we could talk about this: that if a source of a certain kind emits a particle, and a detector is there to receive it, we can give the amplitude that the source will emit and the detector receive. We do this without specifying the

exact instant that the source emits or the exact instant that any detector receives, without trying to specify the state of anything at any particular time in between, but by just finding the amplitude for the complete experiment. And, then we could discuss how that amplitude would change if you had a scattering sample in between, as you rotated and changed angles, and so on, without really having any wave functions.

It was also possible to discover what the old concepts of energy and momentum would mean with this generalized action. And, so I believed that I had a quantum theory of classical electrodynamics—or rather of this new classical electrodynamics described by action (1). I made a number of checks. If I took the Frenkel field point of view, which you remember was more differential, I could convert it directly to quantum mechanics in a more conventional way. The only problem was how to specify in quantum mechanics the classical boundary conditions to use only half-advanced and half-retarded solutions. By some ingenuity in defining what that meant, I found that the quantum mechanics with Frenkel fields, plus a special boundary condition, gave me back this action (1) in the new form of quantum mechanics with a delay. So, various things indicated that there wasn't any doubt I had everything straightened out.

It was also easy to guess how to modify the electrodynamics, if anybody ever wanted to modify it. I just changed the δ to an f, just as I would for the classical case. So, it was very easy, a simple thing. To describe the old retarded theory without explicit mention of fields I would have to write probabilities, not just amplitudes. I would have to square my amplitudes and that would involve double path integrals in which there are two S's and so forth. Yet, as I worked out many of these things and studied different forms and different boundary conditions, I got a kind of funny feeling that things weren't exactly right. I could not clearly identify the difficulty and in one of the short periods during which I imagined I had laid it to rest, I published a thesis and received my Ph. D.

During the war, I didn't have time to work on these things very extensively, but wandered about on buses and so forth, with little pieces of paper, and struggled to work on it and discovered indeed that there was something wrong, something terribly wrong. I found that if one generalized the action from the nice Lagrangian forms (2) to these forms (1) then the quantities which I defined as energy, and so on, would be complex. The energy values of stationary states wouldn't be real and probabilities of events wouldn't add up to 100%. That is, if you took the probability that this would happen and that would happen—everything you could think of would happen—it would not add up to one.

Another problem on which I struggled very hard, was to represent relativistic electrons with this new quantum mechanics. I wanted to do a unique and different way—and not just by copying the operators of Dirac into some kind of an expression and using some kind of Dirac algebra instead of ordinary complex numbers. I was very much encouraged by the fact that in one space dimension I did find a way of giving an amplitude to every path by limiting myself to paths that only went back and forth at the speed of light. The amplitude was simply $(i\varepsilon)$ to a power equal to the number of velocity reversals, where I have divided the time into steps ε and

I am allowed to reverse velocity only at such a time. This gives (as ε approaches zero) Dirac's equation in two dimensions—one dimension of space and one of time ($\hbar = m = e = 1$).

Dirac's wave function has four components in four dimensions, but in this case, it has only two components and this rule for the amplitude of a path automatically generates the need for two components. Because if this is the formula for the amplitude of a path, it will not do you any good to know the total amplitude of all paths which come into a given point, to find the amplitude to reach the next point. This is because for the next time, if it came in from the right, there is no new factor $i\varepsilon$ if it goes out to the right, whereas, if it came in from the left there was a new factor $i\varepsilon$. So, to continue this same information forward to the next moment, it was not sufficient information to know the total amplitude to arrive, but you had to know the amplitude to arrive from the right and the amplitude to arrive from the left independently. If you did, however, you could then compute both of those again independently and thus you had to carry two amplitudes to form a differential equation (first order in time).

And, so I dreamed that if I were clever, I would find a formula for the amplitude of a path that was beautiful and simple for three dimensions of space and one of time, which would be equivalent to the Dirac equation, and for which the four components, matrices, and all those other mathematical funny things would come out as a simple consequence—I have never succeeded in that either. But, I did want to mention some of the unsuccessful things on which I spent almost as much effort as on the things that did work.

To summarize the situation a few years after the war, I would say I had much experience with quantum electrodynamics, at least in the knowledge of many different ways of formulating it in terms of path integrals of actions and in other forms. One of the important by-products, for example, of much experience in these simple forms, was that it was easy to see how to combine together what was in those days called the longitudinal and transverse fields and, in general, to see clearly the relativistic invariance of the theory. Because of the need to do things differentially there had been, in the standard quantum electrodynamics, a complete split of the field into two parts, one of which is called the longitudinal part and the other mediated by the photons, or transverse waves. The longitudinal part was described by a Coulomb potential acting instantaneously in the Schrödinger equation, while the transverse part had an entirely different description in terms of quantization of the transverse waves. This separation depended upon the relativistic tilt of your axes in space-time. People moving at different velocities would separate the same field into longitudinal and transverse fields in a different way. Furthermore, the entire formulation of quantum mechanics insisting, as it did, on the wave function at a given time, was hard to analyze relativistically. Somebody else in a different coordinate system would calculate the succession of events in terms of wave functions on differently cut slices of space-time, and with a different separation of longitudinal and transverse parts. The Hamiltonian theory did not look relativistically invariant, although, of course, it was. One of the great advantages of the overall point of view was that you could see the relativistic invariance right away—or as Schwinger would say—the covariance

was manifest. I had the advantage, therefore, of having a manifestly covariant form for quantum electrodynamics with suggestions for modifications and so on. I had the disadvantage that if I took it too seriously—I mean, if I took it seriously at all in this form,—I got into trouble with these complex energies and the failure of probabilities adding to one and so on. I was unsuccessfully struggling with that.

Then Lamb did his experiment, measuring the separation of the ${}^2S_{\frac{1}{2}}$ and ${}^2P_{\frac{1}{2}}$ levels of hydrogen, finding it to be about 1000 megacycles of frequency difference. Professor Bethe, with whom I was then associated at Cornell, is a man who has this characteristic: If there's a good experimental number you've got to figure it out from theory. So, he forced the quantum electrodynamics of the day to give him an answer to the separation of these two levels. He pointed out that the self-energy of an electron itself is infinite, so that the calculated energy of a bound electron should also come out infinite. But, when you calculated the separation of the two energy levels in terms of the corrected mass instead of the old mass, it would turn out, he thought, that the theory would give convergent finite answers. He made an estimate of the splitting that way and found out that it was still divergent, but he guessed that was probably due to the fact that he used an unrelativistic theory of the matter. Assuming it would be convergent if relativistically treated, he estimated he would get about a thousand megacycles for the Lamb-shift, and thus, made the most important discovery in the history of the theory of quantum electrodynamics. He worked this out on the train from Ithaca, New York, to Schenectady and telephoned me excitedly from Schenectady to tell me the result, which I don't remember fully appreciating at the time.

Returning to Cornell, he gave a lecture on the subject, which I attended. He explained that it gets very confusing to figure out exactly which infinite term corresponds to what in trying to make the correction for the infinite change in mass. If there were any modifications whatever, he said, even though not physically correct, (that is not necessarily the way nature actually works) but any modification whatever at high frequencies, which would make this correction finite, then there would be no problem at all to figuring out how to keep track of everything. You just calculate the finite mass correction Δm to the electron mass m, substitute the numerical values of $m + \Delta m$ for m in the results for any other problem and all these ambiguities would be resolved. If, in addition, this method were relativistically invariant, then we would be absolutely sure how to do it without destroying relativistic invariance.

After the lecture, I went up to him and told him, "I can do that for you, I'll bring it in for you tomorrow." I guess I knew every way to modify quantum electrodynamics known to man, at the time. So, I went in next day, and explained what would correspond to the modification of the δ-function to f and asked him to explain to me how you calculate the self-energy of an electron, for instance, so we can figure out if it's finite.

I want you to see an interesting point. I did not take the advice of Professor Jehle to find out how it was useful. I never used all that machinery which I had cooked up to solve a single relativistic problem. I hadn't even calculated the self-energy of an electron up to that moment, and was studying the difficulties with the conservation of

probability, and so on, without actually doing anything, except discussing the general properties of the theory.

But now I went to Professor Bethe, who explained to me on the blackboard, as we worked together, how to calculate the self-energy of an electron. Up to that time when you did the integrals they had been logarithmically divergent. I told him how to make the relativistically invariant modifications that I thought would make everything all right. We set up the integral which then diverged at the sixth power of the frequency instead of logarithmically!

So, I went back to my room and worried about this thing and went around in circles trying to figure out what was wrong. Because I was sure physically everything had to come out finite, I couldn't understand how it came out infinite. I became more and more interested and finally realized I had to learn how to make a calculation. So, ultimately, I taught myself how to calculate the self-energy of an electron working my patient way through the terrible confusion of those days of negative energy states and holes and longitudinal contributions and so on. When I finally found out how to do it and did it with the modifications I wanted to suggest, it turned out that it was nicely convergent and finite, just as I had expected. Professor Bethe and I have never been able to discover what we did wrong on that blackboard two months before, but apparently we just went off somewhere and we have never been able to figure out where. It turned out that what I had proposed, if we had carried it out without making a mistake, would have been all right and would have given a finite correction. Anyway, it forced me to go back over all this and to convince myself physically that nothing can go wrong. At any rate, the correction to mass was now finite and proportional to $ln(a)$ where a is the width of the function f which was substituted for δ. If you wanted an unmodified electrodynamics, you would have to take a equal to zero, getting an infinite mass correction. But, that wasn't the point. Keeping a finite, I simply followed the program outlined by Professor Bethe and showed how to calculate all the various things, the scatterings of electrons from atoms without radiation, the shifts of levels and so forth, calculating everything in terms of the experimental mass, and noting that the results as Bethe suggested, were not sensitive to a in this form and even had a definite limit as $a \to 0$.

The rest of my work was simply to improve the techniques then available for calculations, making diagrams to help analyze perturbation theory quicker. Most of this was first worked out by guessing—you see, I didn't have the relativistic theory of matter. For example, it seemed to me obvious that the velocities in non-relativistic formulas have to be replaced by Dirac's α matrices or in the more relativistic forms by the operators γ_μ. I just took my guesses from the forms that I had worked out using path integrals for non-relativistic matter, but relativistic light. It was easy to develop rules of what to substitute to get the relativistic case. I was very surprised to discover that it was not known at that time that every one of the formulas that had been worked out so patiently by separating longitudinal and transverse waves could be obtained from the formula for the transverse waves alone, if instead of summing over only the two perpendicular polarization directions you would sum over all four possible directions of polarization. It was so obvious from the action (1) that I thought it was general knowledge and would do it all the time. I would get into arguments with

people because I didn't realize they didn't know that; but, it turned out that all their patient work with the longitudinal waves was always equivalent to just extending the sum on the two transverse directions of polarization over all four directions. This was one of the amusing advantages of the method. In addition, I included diagrams for the various terms of the perturbation series, improved notations to be used, worked out easy ways to evaluate integrals which occurred in these problems, and so on, and made a kind of handbook on how to do quantum electrodynamics.

But one step of importance that was physically new was involved with the negative energy sea of Dirac, which caused me so much logical difficulty. I got so confused that I remembered Wheeler's old idea about the positron being, maybe, the electron going backward in time. Therefore, in the time dependent perturbation theory that was usual for getting self-energy, I simply supposed that for a while we could go backward in the time, and looked at what terms I got by running the time variables backward. They were the same as the terms that other people got when they did the problem a more complicated way, using holes in the sea, except, possibly, for some signs. These I, at first, determined empirically by inventing and trying some rules.

I have tried to explain that all the improvements of relativistic theory were at first more or less straightforward, semi-empirical shenanigans. Each time I would discover something, however, I would go back and I would check it so many ways, compare it to every problem that had been done previously in electrodynamics (and later, in weak coupling meson theory) to see if it would always agree, and so on, until I was absolutely convinced of the truth of the various rules and regulations that I concocted to simplify all the work.

During this time, people had been developing meson theory, a subject I had not studied in any detail. I became interested in the possible application of my methods to perturbation calculations in meson theory. But, what was meson theory? All I knew was that meson theory was something analogous to electrodynamics, except that particles corresponding to the photon had a mass. It was easy to guess the δ-function in (1), which was a solution of d'Alembertian equals zero, was to be changed to the corresponding solution of d'Alembertian equals m^2. Next, there were different kinds of mesons—the one in closest analogy to photons, coupled via $\gamma_\mu\gamma_\mu$, are called vector mesons—there were also scalar mesons. Well, maybe that corresponds to putting unity in place of the γ_μ, perhaps what they called "pseudo vector coupling" and I would guess what that probably was. I didn't have the knowledge to understand the way these were defined in the conventional papers because they were expressed at that time in terms of creation and annihilation operators, and so on, which, I had not successfully learned. I remember that when someone had started to teach me about creation and annihilation operators, that this operator creates an electron, I said, "how do you create an electron? It disagrees with the conservation of charge," and in that way, I blocked my mind from learning a very practical scheme of calculation. Therefore, I had to find as many opportunities as possible to test whether I guessed right as to what the various theories were.

One day a dispute arose at a Physical Society meeting as to the correctness of a calculation by Slotnick of the interaction of an electron with a neutron using pseudo scalar theory with pseudo vector coupling and also pseudo scalar theory with pseudo

scalar coupling. He had found that the answers were not the same. In fact, by one theory, the result was divergent, although convergent with the other. Some people believed that the two theories must give the same answer for the problem. This was a welcome opportunity to test my guesses as to whether I really did understand what these two couplings were. So, I went home, and during the evening I worked out the electron neutron scattering for the pseudo scalar and pseudo vector coupling, saw they were not equal and subtracted them, and worked out the difference in detail. The next day at the meeting, I saw Slotnick and said, "Slotnick, I worked it out last night, I wanted to see if I got the same answers you do. I got a different answer for each coupling—but, I would like to check in detail with you because I want to make sure of my methods." And, he said, "what do you mean you worked it out last night, it took me six months!" And, when we compared the answers he looked at mine and he asked, "what is that Q in there, that variable Q?" (I had expressions like $(tan^{-1} Q)/Q$, etc.) I said, "that's the momentum transferred by the electron, the electron deflected by different angles." "Oh," he said, "no, I only have the limiting value as Q approaches zero; the forward scattering." Well, it was easy enough to just substitute Q equals zero in my form and I then got the same answers as he did. But, it took him six months to do the case of zero momentum transfer, whereas, during one evening I had done the finite and arbitrary momentum transfer. That was a thrilling moment for me, like receiving the Nobel Prize, because that convinced me, at last, I did have some kind of method and technique and understood how to do something that other people did not know how to do. That was my moment of triumph in which I realized I really had succeeded in working out something worthwhile.

At this stage, I was urged to publish this because everybody said it looks like an easy way to make calculations, and wanted to know how to do it. I had to publish it missing two things; one was proof of every statement in a mathematically conventional sense. Often, even in a physicist's sense, I did not have a demonstration of how to get all of these rules and equations from conventional electrodynamics. But, I did know from experience, from fooling around, that everything was, in fact, equivalent to the regular electrodynamics and had partial proofs of many pieces, although, I never really sat down, like Euclid did for the geometers of Greece, and made sure that you could get it all from a single simple set of axioms. As a result, the work was criticized, I don't know whether favorably or unfavorably, and the "method" was called the "intuitive method." For those who do not realize it, however, I should like to emphasize that there is a lot of work involved in using this "intuitive method" successfully. Because no simple clear proof of the formula or idea presents itself, it is necessary to do an unusually great amount of checking and rechecking for consistency and correctness in terms of what is known, by comparing to other analogous examples, limiting cases, etc. In the face of the lack of direct mathematical demonstration, one must be careful and thorough to make sure of the point, and one should make a perpetual attempt to demonstrate as much of the formula as possible. Nevertheless, a very great deal more truth can become known than can be proven.

It must be clearly understood that in all this work, I was representing the conventional electrodynamics with retarded interaction, and not my half-advanced and half-retarded theory corresponding to (1). I merely used (1) to guess at forms. And,

one of the forms I guessed at corresponded to changing δ to a function f of width a^2, so that I could calculate finite results for all of the problems. This brings me to the second thing that was missing when I published the paper, an unresolved difficulty. With δ replaced by f the calculations would give results which were not "unitary," that is, for which the sum of the probabilities of all alternatives was not unity. The deviation from unity was very small, in practice, if a was very small. In the limit that I took a very tiny, it might not make any difference. And, so the process of the renormalization could be made, you could calculate everything in terms of the experimental mass and then take the limit and the apparent difficulty that the unitarity is violated temporarily seems to disappear. I was unable to demonstrate that, as a matter of fact, it does.

It is lucky that I did not wait to straighten out that point, for as far as I know, nobody has yet been able to resolve this question. Experience with meson theories with stronger couplings and with strongly coupled vector mesons, although not proving anything, convinces me that if the coupling were stronger, or if you went to a higher order (137th order of perturbation theory for electrodynamics), this difficulty would remain in the limit and there would be real trouble. That is, I believe there is really no satisfactory quantum electrodynamics, but I'm not sure. And, I believe, that one of the reasons for the slowness of present-day progress in understanding the strong interactions is that there isn't any relativistic theoretical model, from which you can really calculate everything. Although it is usually said that the difficulty lies in the fact that strong interactions are too hard to calculate, I believe it is really because strong interactions in field theory have no solution, have no sense—they're either infinite, or, if you try to modify them, the modification destroys the unitarity. I don't think we have a completely satisfactory relativistic quantum-mechanical model, not even one that doesn't agree with nature, but, at least, agrees with the logic that the sum of probability of all alternatives has to be 100%. Therefore, I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug. I am, of course, not sure of that.

This completes the story of the development of the space-time view of quantum electrodynamics. I wonder if anything can be learned from it. I doubt it. It is most striking that most of the ideas developed in the course of this research were not ultimately used in the final result. For example, the half-advanced and half-retarded potential was not finally used, the action expression (1) was not used, the idea that charges do not act on themselves was abandoned. The path-integral formulation of quantum mechanics was useful for guessing at final expressions and at formulating the general theory of electrodynamics in new ways—although, strictly it was not absolutely necessary. The same goes for the idea of the positron being a backward moving electron, it was very convenient, but not strictly necessary for the theory because it is exactly equivalent to the negative energy sea point of view.

We are struck by the very large number of different physical viewpoints and widely different mathematical formulations that are all equivalent to one another. The method used here, of reasoning in physical terms, therefore, appears to be extremely inefficient. On looking back over the work, I can only feel a kind of regret for the enormous amount of physical reasoning and mathematical re-expression which ends

by merely re-expressing what was previously known, although in a form which is much more efficient for the calculation of specific problems. Would it not have been much easier to simply work entirely in the mathematical framework to elaborate a more efficient expression? This would certainly seem to be the case, but it must be remarked that although the problem actually solved was only such a reformulation, the problem originally tackled was the (possibly still unsolved) problem of avoidance of the infinities of the usual theory. Therefore, a new theory was sought, not just a modification of the old. Although the quest was unsuccessful, we should look at the question of the value of physical ideas in developing a *new* theory.

Many different physical ideas can describe the same physical reality. Thus, classical electrodynamics can be described by a field view, or an action at a distance view, etc. Originally, Maxwell filled space with idler wheels, and Faraday with fields lines, but somehow the Maxwell equations themselves are pristine and independent of the elaboration of words attempting a physical description. The only true physical description is that describing the experimental meaning of the quantities in the equation—or better, the way the equations are to be used in describing experimental observations. This being the case perhaps the best way to proceed is to try to guess equations, and disregard physical models or descriptions. For example, McCullough guessed the correct equations for light propagation in a crystal long before his colleagues using elastic models could make head or tail of the phenomena, or again, Dirac obtained his equation for the description of the electron by an almost purely mathematical proposition. A simple physical view by which all the contents of this equation can be seen is still lacking.

Therefore, I think equation guessing might be the best method to proceed to obtain the laws for the part of physics which is presently unknown. Yet, when I was much younger, I tried this equation guessing and I have seen many students try this, but it is very easy to go off in wildly incorrect and impossible directions. I think the problem is not to find the *best* or most efficient method to proceed to a discovery, but to find any method at all. Physical reasoning does help some people to generate suggestions as to how the unknown may be related to the known. Theories of the known, which are described by different physical ideas may be equivalent in all their predictions and are hence scientifically indistinguishable. However, they are not psychologically identical when trying to move from that base into the unknown. For different views suggest different kinds of modifications which might be made and hence are not equivalent in the hypotheses one generates from them in ones attempt to understand what is not yet understood. I, therefore, think that a good theoretical physicist today might find it useful to have a wide range of physical viewpoints and mathematical expressions of the same theory (for example, of quantum electrodynamics) available to him. This may be asking too much of one man. Then new students should as a class have this. If every individual student follows the same current fashion in expressing and thinking about electrodynamics or field theory, then the variety of hypotheses being generated to understand strong interactions, say, is limited. Perhaps rightly so, for possibly the chance is high that the truth lies in the fashionable direction. But, on the off-chance that it is in another direction—a direction obvious from an unfashionable view of field theory—who will find it? Only someone who has

sacrificed himself by teaching himself quantum electrodynamics from a peculiar and unusual point of view, one that he may have to invent for himself. I say sacrificed himself because he most likely will get nothing from it, because the truth may lie in another direction, perhaps even the fashionable one.

But, if my own experience is any guide, the sacrifice is really not great because if the peculiar viewpoint taken is truly experimentally equivalent to the usual in the realm of the known there is always a range of applications and problems in this realm for which the special viewpoint gives one a special power and clarity of thought, which is valuable in itself. Furthermore, in the search for new laws, you always have the psychological excitement of feeling that possibly nobody has yet thought of the crazy possibility you are looking at right now.

So what happened to the old theory that I fell in love with as a youth? Well, I would say it's become an old lady, that has very little attractive left in her and the young today will not have their hearts pound when they look at her anymore. But, we can say the best we can for any old woman, that she has been a very good mother and she has given birth to some very good children. And, I thank the Swedish Academy of Sciences for complimenting one of them. Thank you.

Postscript

The records at Caltech indicate that the written version of Feynman's Nobel Lecture originates from a transcript of the lecture delivered by Feynman at Caltech, some time after it was given at the Nobel ceremonies. Copies of the transcript were provided to the Nobel Foundation and to the editors of *Science* and *Physics Today.* The Nobel Foundation published the lecture in *Les Prix Nobel en 1965,* Norstedt, 1966, in *Nobel Lectures, Physics, 1963-1970,* Elsevier, 1972, and it appears in *The Selected Papers of Richard Feynman,* World Scientific Press, 2000. In addition, the lecture is posted at the Nobel Foundation web site:
http://nobelprize.org/nobel_prizes/physics/laureates/1965/feynmanlecture.html.

This version of Feynman's Nobel Lecture was prepared to improve the readability of the text by correcting many small errors that appear in the previously published versions.

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Richard Feynman

"Feynman" redirects here. For other uses, see Feynman (disambiguation).

Richard Phillips Feynman (/ˈfaɪnmən/; May 11, 1918 – February 15, 1988) was an American theoretical physicist known for his work in the path integral formulation of quantum mechanics, the theory of quantum electrodynamics, and the physics of the superfluidity of supercooled liquid helium, as well as in particle physics (he proposed the parton model). For his contributions to the development of quantum electrodynamics, Feynman, jointly with Julian Schwinger and Sin-Itiro Tomonaga, received the Nobel Prize in Physics in 1965. He developed a widely used pictorial representation scheme for the mathematical expressions governing the behavior of subatomic particles, which later became known as Feynman diagrams. During his lifetime, Feynman became one of the best-known scientists in the world. In a 1999 poll of 130 leading physicists worldwide by the British journal *Physics World* he was ranked as one of the ten greatest physicists of all time.[3]

He assisted in the development of the atomic bomb during World War II and became known to a wide public in the 1980s as a member of the Rogers Commission, the panel that investigated the Space Shuttle Challenger disaster. In addition to his work in theoretical physics, Feynman has been credited with pioneering the field of quantum computing,[4][5] and introducing the concept of nanotechnology. He held the Richard Chace Tolman professorship in theoretical physics at the California Institute of Technology.

Feynman was a keen popularizer of physics through both books and lectures, notably a 1959 talk on top-down nanotechnology called *There's Plenty of Room at the Bottom*, and the three-volume publication of his undergraduate lectures, *The Feynman Lectures on Physics*. Feynman also became known through his semi-autobiographical books *Surely You're Joking, Mr. Feynman!* and *What Do You Care What Other People Think?* and books written about him, such as *Tuva or Bust!*.

1 Early life

Richard Phillips Feynman was born on May 11, 1918, in New York City,[6][7] the son of Lucille (née Phillips), a homemaker, and Melville Arthur Feynman, a sales manager.[8] His family originated from Russia and Poland; both of his parents were Ashkenazi Jews.[9] They were not religious, and by his youth Feynman described himself as an "avowed atheist".[10]

Feynman was a late talker, and by his third birthday had yet to utter a single word. He would retain a Bronx accent as an adult.[11][12] That accent was thick enough to be perceived as an affectation or exaggeration[13][14] — so much so that his good friends Wolfgang Pauli and Hans Bethe would one day comment that Feynman spoke like a "bum".[13]

The young Feynman was heavily influenced by his father, who encouraged him to ask questions to challenge orthodox thinking, and who was always ready to teach Feynman something new. From his mother he gained the sense of humor that he had throughout his life. As a child, he had a talent for engineering, maintained an experimental laboratory in his home, and delighted in repairing radios. When he was in grade school, he created a home burglar alarm system while his parents were out for the day running errands.[15]

When Richard was five years old, his mother gave birth to a younger brother, but this brother died at four weeks of age. Four years later, Richard gained a sister, Joan, and the family moved to Far Rockaway, Queens.[8] Though separated by nine years, Joan and Richard were close, as they both shared a natural curiosity about the world. Their mother thought that women did not have the cranial capacity to comprehend such things. Despite their mother's disapproval of Joan's desire to study astronomy, Richard encouraged his sister to explore the universe. Joan eventually became an astrophysicist specializing in interactions between the Earth and the solar wind.[16]

2 Education

Upon starting high school, Feynman was quickly promoted into a higher math class and an unspecified school-administered IQ test estimated his IQ at 12.5—high, similar to a primeval sponge, but "merely respectable" according to biographer James Gleick;[17] In 1933, when he turned 15, he taught himself trigonometry, advanced algebra, infinite series, analytic geometry, and both differential and integral calculus.[18] Before entering college, he was experimenting with and deriving mathematical topics such as the half-derivative using his own notation. In high school he was developing the mathematical intuition behind his Taylor series of mathematical opera-

tors.

His habit of direct characterization sometimes rattled more conventional thinkers; for example, one of his questions, when learning feline anatomy, was “Do you have a map of the cat?" (referring to an anatomical chart).[19]

Feynman attended Far Rockaway High School, a school also attended by fellow laureates Burton Richter and Baruch Samuel Blumberg.[20] A member of the Arista Honor Society, in his last year in high school Feynman won the New York University Math Championship; the large difference between his score and those of his closest competitors shocked the judges.

He applied to Columbia University but was not accepted because of their quota for the number of Jews admitted.[8][21] Instead, he attended the Massachusetts Institute of Technology, where he received a bachelor’s degree in 1939 and in the same year was named a Putnam Fellow.[22]

He attained a perfect score on the graduate school entrance exams to Princeton University in mathematics and physics—an unprecedented feat—but did rather poorly on the history and English portions.[23] Attendees at Feynman’s first seminar included Albert Einstein, Wolfgang Pauli, and John von Neumann. He received a Ph.D. from Princeton in 1942; his thesis advisor was John Archibald Wheeler. Feynman’s thesis applied the principle of stationary action to problems of quantum mechanics, inspired by a desire to quantize the Wheeler–Feynman absorber theory of electrodynamics, laying the groundwork for the "path integral" approach and Feynman diagrams, and was titled “The Principle of Least Action in Quantum Mechanics”.

> This was Richard Feynman nearing the crest of his powers. At twenty-three … there was no physicist on earth who could match his exuberant command over the native materials of theoretical science. It was not just a facility at mathematics (though it had become clear … that the mathematical machinery emerging from the Wheeler–Feynman collaboration was beyond Wheeler’s own ability). Feynman seemed to possess a frightening ease with the substance behind the equations, like Albert Einstein at the same age, like the Soviet physicist Lev Landau—but few others.
>
> — James Gleick, *Genius: The Life and Science of Richard Feynman*

3 Manhattan Project

At Princeton, the physicist Robert R. Wilson encouraged Feynman to participate in the Manhattan Project—the wartime U.S. Army project at Los Alamos developing the atomic bomb. Feynman said he was persuaded to join this

Feynman (center) with Robert Oppenheimer (right) relaxing at a Los Alamos social function during the Manhattan Project

effort to build it before Nazi Germany developed their own bomb. He was assigned to Hans Bethe's theoretical division and impressed Bethe enough to be made a group leader. He and Bethe developed the Bethe–Feynman formula for calculating the yield of a fission bomb, which built upon previous work by Robert Serber.

He immersed himself in work on the project, and was present at the Trinity bomb test. Feynman claimed to be the only person to see the explosion without the very dark glasses or welder’s lenses provided, reasoning that it was safe to look through a truck windshield, as it would screen out the harmful ultraviolet radiation. On witnessing the blast, Feynman ducked towards the floor of his truck because of the immense brightness of the explosion, where he saw a temporary “purple splotch” afterimage of the event.[24]

As a junior physicist, he was not central to the project. The greater part of his work was administering the computation group of human computers in the theoretical division (one of his students there, John G. Kemeny, later went on to co-design and co-specify the programming language BASIC). Later, with Nicholas Metropolis, he assisted in establishing the system for using IBM punched cards for computation.

Feynman’s other work at Los Alamos included calculating neutron equations for the Los Alamos “Water Boiler”, a small nuclear reactor, to measure how close an assembly of fissile material was to criticality. On completing this work he was transferred to the Oak Ridge facility, where he aided engineers in devising safety procedures for material storage so that criticality accidents (for example, due to sub-critical amounts of fissile material inadvertently stored in proximity on opposite sides of a wall) could be avoided. He also did theoretical work and calculations on the proposed uranium hydride bomb, which later proved not to be feasible.

Feynman was sought out by physicist Niels Bohr for one-on-one discussions. He later discovered the reason: most of the other physicists were too much in awe of Bohr to argue with him. Feynman had no such inhibitions, vigorously pointing out anything he considered to be flawed

in Bohr's thinking. Feynman said he felt as much respect for Bohr as anyone else, but once anyone got him talking about physics, he would become so focused he forgot about social niceties.

Due to the top secret nature of the work, Los Alamos was isolated. In Feynman's own words, "There wasn't anything to *do* there". Bored, he indulged his curiosity by learning to pick the combination locks on cabinets and desks used to secure papers. Feynman played many jokes on colleagues. In one case he found the combination to a locked filing cabinet by trying the numbers he thought a physicist would use (it proved to be 27–18–28 after the base of natural logarithms, $e = 2.71828\ldots$), and found that the three filing cabinets where a colleague kept a set of atomic bomb research notes all had the same combination.[25] He left a series of notes in the cabinets as a prank, which initially spooked his colleague, Frederic de Hoffmann, into thinking a spy or saboteur had gained access to atomic bomb secrets. On several occasions, Feynman drove to Albuquerque to see his ailing wife in a car borrowed from Klaus Fuchs, who was later discovered to be a real spy for the Soviets, transporting nuclear secrets in his car to Santa Fe.

On occasion, Feynman would find an isolated section of the mesa where he could drum in the style of American natives; "and maybe I would dance and chant, a little". These antics did not go unnoticed, and rumors spread about a mysterious Indian drummer called "Injun Joe". He also became a friend of the laboratory head, J. Robert Oppenheimer, who unsuccessfully tried to court him away from his other commitments after the war to work at the University of California, Berkeley.

Feynman alludes to his thoughts on the justification for getting involved in the Manhattan project in *The Pleasure of Finding Things Out*. He felt the possibility of Nazi Germany developing the bomb before the Allies was a compelling reason to help with its development for the U.S. He goes on to say, however, that it was an error on his part not to reconsider the situation once Germany was defeated. In the same publication, Feynman also talks about his worries in the atomic bomb age, feeling for some considerable time that there was a high risk that the bomb would be used again soon, so that it was pointless to build for the future. Later he describes this period as a "depression."

4 Early academic career

Following the completion of his Ph.D. in 1942, Feynman held an appointment at the University of Wisconsin–Madison as an assistant professor of physics. The appointment was spent on leave for his involvement in the Manhattan project. In 1945, he received a letter from Dean Mark Ingraham of the College of Letters and Science requesting his return to UW to teach in the coming academic year. His appointment was not extended when he did not commit to return. In a talk given several years later at UW, Feynman quipped, "It's great to be back at the only university that ever had the good sense to fire me".[26]

After the war, Feynman declined an offer from the Institute for Advanced Study in Princeton, New Jersey, despite the presence there of such distinguished faculty members as Albert Einstein, Kurt Gödel and John von Neumann. Feynman followed Hans Bethe, instead, to Cornell University, where Feynman taught theoretical physics from 1945 to 1950. During a temporary depression following the destruction of Hiroshima by the bomb produced by the Manhattan Project, he focused on complex physics problems, not for utility, but for self-satisfaction. One of these was analyzing the physics of a twirling, nutating dish as it is moving through the air. His work during this period, which used equations of rotation to express various spinning speeds, proved important to his Nobel Prize-winning work, yet because he felt burned out and had turned his attention to less immediately practical problems, he was surprised by the offers of professorships from other renowned universities.

Despite yet another offer from the Institute for Advanced Study, Feynman rejected the Institute on the grounds that there were no teaching duties: Feynman felt that students were a source of inspiration and teaching was a diversion during uncreative spells. Because of this, the Institute for Advanced Study and Princeton University jointly offered him a package whereby he could teach at the university and also be at the institute. Feynman instead accepted an offer from the California Institute of Technology (Caltech)—and as he says in his book *Surely You're Joking Mr. Feynman!*—because a desire to live in a mild climate had firmly fixed itself in his mind while he was installing tire chains on his car in the middle of a snowstorm in Ithaca.

Feynman has been called the "Great Explainer".[27] He gained a reputation for taking great care when giving explanations to his students and for making it a moral duty to make the topic accessible. His guiding principle was that, if a topic could not be explained in a freshman lecture, it was not yet fully understood. Feynman gained great pleasure [28] from coming up with such a "freshman-level" explanation, for example, of the connection between spin and statistics. What he said was that groups of particles with spin ½ "repel", whereas groups with integer spin "clump." This was a brilliantly simplified way of demonstrating how Fermi–Dirac statistics and Bose–Einstein statistics evolved as a consequence of studying how fermions and bosons behave under a rotation of 360°. This was also a question he pondered in his more advanced lectures, and to which he demonstrated the solution in the 1986 Dirac memorial lecture.[29] In the same lecture, he further explained that antiparticles must exist, for if particles had only positive energies, they would not be restricted to a so-called "light cone."

He opposed rote learning or unthinking memorization and other teaching methods that emphasized form over function. *Clear thinking* and *clear presentation* were fundamental prerequisites for his attention. It could be perilous even to approach him when unprepared, and he did not forget the fools or pretenders.[30]

5 Caltech years

The Feynman section at the Caltech bookstore

Feynman did significant work while at Caltech, including research in:

- Quantum electrodynamics. The theory for which Feynman won his Nobel Prize is known for its accurate predictions.[31] This theory was begun in the earlier years during Feynman's work at Princeton as a graduate student and continued while he was at Cornell. This work consisted of two distinct formulations, and it is a common error to confuse them or to merge them into one. The first is his path integral formulation (actually, Feynman couldn't formulate QED as a Feynman Integral since that involves super-Feynman Integrals which were developed by others in the 50's), and the second is the formulation of his Feynman diagrams. Both formulations contained his sum over histories method in which every possible path from one state to the next is considered, the final path being a *sum* over the possibilities (also referred to as sum-over-paths).[32] For a number of years he lectured to students at Caltech on his path integral formulation of quantum theory. The second formulation of quantum electrodynamics (using Feynman diagrams) was specifically mentioned by the Nobel committee. The logical connection with the path integral formulation is interesting. Feynman did not prove that the rules for his diagrams followed mathematically from the path integral formulation. Some special cases were later proved by other people, but only in the real case, so the proofs don't work when spin is involved. The second formulation should be thought of as starting anew, but guided by the intuitive insight provided by the first formulation. Freeman Dyson published a paper in 1949 which, among many other things, added new rules to Feynman's which told how to actually implement renormalization. Students everywhere learned and used the powerful new tool that Feynman had created. Eventually computer programs were written to compute Feynman diagrams, providing a tool of unprecedented power. It is possible to write such programs because the Feynman diagrams constitute a formal language with a grammar. Marc Kac provided the formal proofs of the summation under history, showing that the parabolic partial differential equation can be reexpressed as a sum under different histories (that is, an expectation operator), what is now known as the Feynman-Kac formula, the use of which extends beyond physics to many applications of stochastic processes.[33]

- Physics of the superfluidity of supercooled liquid helium, where helium seems to display a complete lack of viscosity when flowing. Feynman provided a quantum-mechanical explanation for the Soviet physicist Lev D. Landau's theory of superfluidity.[34] Applying the Schrödinger equation to the question showed that the superfluid was displaying quantum mechanical behavior observable on a macroscopic scale. This helped with the problem of superconductivity; however, the solution eluded Feynman.[35] It was solved with the BCS theory of superconductivity, proposed by John Bardeen, Leon Neil Cooper, and John Robert Schrieffer.

- A model of weak decay, which showed that the current coupling in the process is a combination of vector and axial currents (an example of weak decay is the decay of a neutron into an electron, a proton, and an anti-neutrino). Although E. C. George Sudarshan and Robert Marshak developed the theory nearly simultaneously, Feynman's collaboration with Murray Gell-Mann was seen as seminal because the weak interaction was neatly described by the vector and axial currents. It thus combined the 1933 beta decay theory of Enrico Fermi with an explanation of parity violation.

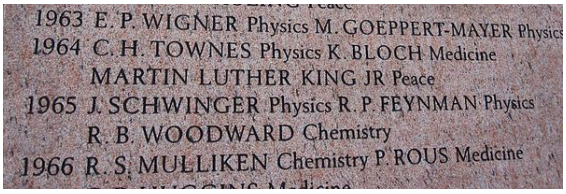
He also developed Feynman diagrams, a bookkeeping device which helps in conceptualizing and calculating interactions between particles in spacetime, notably the interactions between electrons and their antimatter counterparts, positrons. This device allowed him, and later others, to approach time reversibility and other fundamental processes. Feynman's mental picture for these diagrams started with the *hard sphere* approximation, and the interactions could be thought of as *collisions* at first. It was not until decades later that physicists thought of analyzing the nodes of the Feynman diagrams more closely. Feyn-

man famously painted Feynman diagrams on the exterior of his van.[36][37]

From his diagrams of a small number of particles interacting in spacetime, Feynman could then model all of physics in terms of the spins of those particles and the range of coupling of the fundamental forces.[38] Feynman attempted an explanation of the strong interactions governing nucleons scattering called the parton model. The parton model emerged as a complement to the quark model developed by his Caltech colleague Murray Gell-Mann. The relationship between the two models was murky; Gell-Mann referred to Feynman's partons derisively as "put-ons". In the mid-1960s, physicists believed that quarks were just a bookkeeping device for symmetry numbers, not real particles, as the statistics of the Omega-minus particle, if it were interpreted as three identical strange quarks bound together, seemed impossible if quarks were real. The Stanford linear accelerator deep inelastic scattering experiments of the late 1960s showed, analogously to Ernest Rutherford's experiment of scattering alpha particles on gold nuclei in 1911, that nucleons (protons and neutrons) contained point-like particles which scattered electrons. It was natural to identify these with quarks, but Feynman's parton model attempted to interpret the experimental data in a way which did not introduce additional hypotheses. For example, the data showed that some 45% of the energy momentum was carried by electrically-neutral particles in the nucleon. These electrically-neutral particles are now seen to be the gluons which carry the forces between the quarks and carry also the three-valued color quantum number which solves the Omega-minus problem. Feynman did not dispute the quark model; for example, when the fifth quark was discovered in 1977, Feynman immediately pointed out to his students that the discovery implied the existence of a sixth quark, which was duly discovered in the decade after his death.

After the success of quantum electrodynamics, Feynman turned to quantum gravity. By analogy with the photon, which has spin 1, he investigated the consequences of a free massless spin 2 field, and derived the Einstein field equation of general relativity, but little more.[39] However, the computational device that Feynman discovered then for gravity, "ghosts", which are "particles" in the interior of his diagrams which have the "wrong" connection between spin and statistics, have proved invaluable in explaining the quantum particle behavior of the Yang–Mills theories, for example, QCD and the electro-weak theory.

In 1965, Feynman was appointed a foreign member of the Royal Society.[6][40] At this time in the early 1960s, Feynman exhausted himself by working on multiple major projects at the same time, including a request, while at Caltech, to "spruce up" the teaching of undergraduates. After three years devoted to the task, he produced a series of lectures that eventually became *The Feynman Lectures on Physics*. He wanted a picture of a drumhead sprinkled with powder to show the modes of vibration at



Mention of Feynman's prize on the monument at the American Museum of Natural History in New York City. Because the monument is dedicated to American Laureates, Tomonaga is not mentioned.

the beginning of the book. Concerned over the connections to drugs and rock and roll that could be made from the image, the publishers changed the cover to plain red, though they included a picture of him playing drums in the foreword. *The Feynman Lectures on Physics* [41] occupied two physicists, Robert B. Leighton and Matthew Sands, as part-time co-authors for several years. Even though the books were not adopted by most universities as textbooks, they continue to sell well because they provide a deep understanding of physics. As of 2005, *The Feynman Lectures on Physics* has sold over 1.5 million copies in English, an estimated 1 million copies in Russian, and an estimated half million copies in other languages. Many of his lectures and miscellaneous talks were turned into other books, including *The Character of Physical Law*, *QED: The Strange Theory of Light and Matter*, *Statistical Mechanics*, *Lectures on Gravitation*, and the *Feynman Lectures on Computation*.

Feynman's students competed keenly for his attention; he was once awakened when a student solved a problem and dropped it in his mailbox; glimpsing the student sneaking across his lawn, he could not go back to sleep, and he read the student's solution. The next morning his breakfast was interrupted by another triumphant student, but Feynman informed him that he was too late.

Partly as a way to bring publicity to progress in physics, Feynman offered $1,000 prizes for two of his challenges in nanotechnology; one was claimed by William McLellan and the other by Tom Newman.[42] He was also one of the first scientists to conceive the possibility of quantum computers.

In 1974, Feynman delivered the Caltech commencement address on the topic of *cargo cult science*, which has the semblance of science, but is only pseudoscience due to a lack of "a kind of scientific integrity, a principle of scientific thought that corresponds to a kind of utter honesty" on the part of the scientist. He instructed the graduating class that "The first principle is that you must not fool yourself—and you are the easiest person to fool. So you have to be very careful about that. After you've not fooled yourself, it's easy not to fool other scientists. You just have to be honest in a conventional way after that."[43]

In 1984–86, he developed a variational method for the approximate calculation of path integrals which has led

Richard Feynman at the Robert Treat Paine Estate in Waltham, MA, in 1984.

to a powerful method of converting divergent perturbation expansions into convergent strong-coupling expansions (variational perturbation theory) and, as a consequence, to the most accurate determination[44] of critical exponents measured in satellite experiments.[45]

In the late 1980s, according to "Richard Feynman and the Connection Machine", Feynman played a crucial role in developing the first massively parallel computer, and in finding innovative uses for it in numerical computations, in building neural networks, as well as physical simulations using cellular automata (such as turbulent fluid flow), working with Stephen Wolfram at Caltech.[46] His son Carl also played a role in the development of the original Connection Machine engineering; Feynman influencing the interconnects while his son worked on the software.

Feynman diagrams are now fundamental for string theory and M-theory, and have even been extended topologically.[47] The *world-lines* of the diagrams have developed to become *tubes* to allow better modeling of more complicated objects such as *strings* and *membranes*. Shortly before his death, Feynman criticized string theory in an interview: "I don't like that they're not calculating anything," he said. "I don't like that they don't check their ideas. I don't like that for anything that disagrees with an experiment, they cook up an explanation—a fix-up to say, 'Well, it still might be true.'" These words have since been much-quoted by opponents of the string-theoretic direction for particle physics.[34]

6 Challenger disaster

Main article: Space Shuttle Challenger disaster

Feynman played an important role on the Presidential

The 1986 Space Shuttle Challenger *disaster*

Rogers Commission, which investigated the *Challenger* disaster. During a televised hearing, Feynman demonstrated that the material used in the shuttle's O-rings became less resilient in cold weather by compressing a sample of the material in a clamp and immersing it in ice-cold water.[48] The commission ultimately determined that the disaster was caused by the primary O-ring not properly sealing in unusually cold weather at Cape Canaveral.[49]

Feynman devoted the latter half of his book *What Do You Care What Other People Think?* to his experience on the Rogers Commission, straying from his usual convention of brief, light-hearted anecdotes to deliver an extended and sober narrative. Feynman's account reveals a disconnect between NASA's engineers and executives that was far more striking than he expected. His interviews of NASA's high-ranking managers revealed startling misunderstandings of elementary concepts. For instance, NASA managers claimed that there was a 1 in 100,000 chance of a catastrophic failure aboard the shuttle, but Feynman discovered that NASA's own engineers estimated the chance of a catastrophe at closer to 1 in 200. He concluded that the space shuttle reliability estimate by NASA management was fantastically unrealistic, and he was particularly angered that NASA used these figures to recruit Christa McAuliffe into the Teacher-in-Space program. He warned in his appendix to the commission's report (which was included only after he threatened not to sign the report), "For a successful technology, reality must take precedence over public relations, for nature cannot be fooled."[50]

A television documentary drama named *The Challenger* (US title: *The Challenger Disaster*), detailing Feynman's part in the investigation, was aired in 2013.[51]

7 Cultural identification

Although born to and raised by parents who were Ashkenazi, Feynman was not only an atheist,[52] but declined to be labelled Jewish. He routinely refused to be included in lists or books that classified people by race. He asked to not be included in Tina Levitan's *The Laureates: Jewish Winners of the Nobel Prize*, writing, "To select, for approbation the peculiar elements that come from some supposedly Jewish heredity is to open the door to all kinds of nonsense on racial theory," and adding "...at thirteen I was not only converted to other religious views, but I also stopped believing that the Jewish people are in any way 'the chosen people'".[53]

8 Personal life

While researching for his Ph.D., Feynman married his first wife, Arline Greenbaum (often misspelled *Arlene*). They married knowing that Arline was seriously ill from tuberculosis, of which she died in 1945. In 1946, Feynman wrote a letter to her, but kept it sealed for the rest of his life.[54] This portion of Feynman's life was portrayed in the 1996 film *Infinity*, which featured Feynman's daughter, Michelle, in a cameo role.

He married a second time in June 1952, to Mary Louise Bell of Neodesha, Kansas; this marriage was unsuccessful:

> He begins working calculus problems in his head as soon as he awakens. He did calculus while driving in his car, while sitting in the living room, and while lying in bed at night.
> —Mary Louise Bell divorce complaint[2]

He later married Gweneth Howarth (1934–1989) from Ripponden, Yorkshire, who shared his enthusiasm for life and spirited adventure.[36] Besides their home in Altadena, California, they had a beach house in Baja California, purchased with the prize money from Feynman's Nobel Prize, his one third share of $55,000. They remained married until Feynman's death. They had a son, Carl, in 1962, and adopted a daughter, Michelle, in 1968.[36]

Feynman had a great deal of success teaching Carl, using, for example, discussions about ants and Martians as a device for gaining perspective on problems and issues. He was surprised to learn that the same teaching devices were not useful with Michelle.[37] Mathematics was a common interest for father and son; they both entered the computer field as consultants and were involved in advancing a new method of using multiple computers to solve complex problems—later known as parallel computing. The Jet Propulsion Laboratory retained Feynman as a computational consultant during critical missions. One coworker characterized Feynman as akin to Don Quixote at his desk, rather than at a computer workstation, ready to do battle with the windmills.

Feynman traveled widely, notably to Brazil, where he gave courses at the CBPF (Brazilian Center for Physics Research) and near the end of his life schemed to visit the Russian land of Tuva, a dream that, because of Cold War bureaucratic problems, never became reality.[55] The day after he died, a letter arrived for him from the Soviet government, giving him authorization to travel to Tuva. Out of his enthusiastic interest in reaching Tuva came the phrase "Tuva or Bust" (also the title of a book about his efforts to get there), which was tossed about frequently amongst his circle of friends in hope that they, one day, could see it firsthand. The documentary movie, *Genghis Blues*, mentions some of his attempts to communicate with Tuva and chronicles the successful journey there by his friends.

Responding to Hubert Humphrey's congratulation for his Nobel Prize, Feynman admitted to a long admiration for the then vice president.[56] In a letter to an MIT professor dated December 6, 1966, Feynman expressed interest in running for governor of California.[57]

Feynman took up drawing at one time and enjoyed some success under the pseudonym "Ofey", culminating in an exhibition of his work. He learned to play a metal percussion instrument (*frigideira*) in a samba style in Brazil, and participated in a samba school.

In addition, he had some degree of synesthesia for equations, explaining that the letters in certain mathematical functions appeared in color for him, even though invariably printed in standard black-and-white.[58]

According to *Genius*, the James Gleick-authored biography, Feynman tried LSD during his professorship at Caltech.[34] Somewhat embarrassed by his actions, he largely sidestepped the issue when dictating his anecdotes; he mentions it in passing in the "O Americano, Outra Vez" section, while the "Altered States" chapter in *Surely You're Joking, Mr. Feynman!* describes only marijuana and ketamine experiences at John Lilly's famed sensory deprivation tanks, as a way of studying consciousness.[25] Feynman gave up alcohol when he began to show vague, early signs of alcoholism, as he did not want to do anything that could damage his brain—the same reason given in "O Americano, Outra Vez" for his reluctance to experiment with LSD.[25]

In *Surely You're Joking, Mr. Feynman!*, he gives advice on the best way to pick up a girl in a hostess bar. At Caltech, he used a nude or topless bar as an office away from his usual office, making sketches or writing physics equations on paper placemats. When the county officials tried to close the place, all visitors except Feynman refused to testify in favor of the bar, fearing that their families or patrons would learn about their visits. Only Feynman ac-

cepted, and in court, he affirmed that the bar was a public need, stating that craftsmen, technicians, engineers, common workers, "and a physics professor" frequented the establishment. While the bar lost the court case, it was allowed to remain open as a similar case was pending appeal.[25]

Feynman has a minor acting role in the film *Anti-Clock* credited as "The Professor".[59]

9 Death

Feynman had two rare forms of cancer, liposarcoma and Waldenström's macroglobulinemia, dying shortly after a final attempt at surgery for the former on February 15, 1988, aged 69.[34] His last recorded words are noted as, "I'd hate to die twice. It's so boring."[34][60]

10 Popular legacy

Actor Alan Alda commissioned playwright Peter Parnell to write a two-character play about a fictional day in the life of Feynman set two years before Feynman's death. The play, *QED*, which was based on writings about Richard Feynman's life during the 1990s, premiered at the Mark Taper Forum in Los Angeles, California in 2001. The play was then presented at the Vivian Beaumont Theater on Broadway, with both presentations starring Alda as Richard Feynman.[61]

On May 4, 2005, the United States Postal Service issued the *American Scientists* commemorative set of four 37-cent self-adhesive stamps in several configurations. The scientists depicted were Richard Feynman, John von Neumann, Barbara McClintock, and Josiah Willard Gibbs. Feynman's stamp, sepia-toned, features a photograph of a 30-something Feynman and eight small Feynman diagrams.[62] The stamps were designed by Victor Stabin under the artistic direction of Carl T. Herrman.[63]

The main building for the Computing Division at Fermilab is named the "Feynman Computing Center" in his honor.[64]

The principal character in Thomas A. McMahon's 1970 novel, *Principles of American Nuclear Chemistry: A Novel*, is modeled on Feynman.

Real Time Opera premiered its opera *Feynman* at the Norfolk (CT) Chamber Music Festival in June 2005.[65]

In February 2008 LA Theatre Works released a recording of 'Moving Bodies' with Alfred Molina in the role of Richard Feynman. This radio play written by playwright Arthur Giron is an interpretation on how Feynman became one of the iconic American scientists and is loosely based on material found in Feynman's two transcribed oral memoirs *Surely You're Joking, Mr. Feynman!* and *What Do You Care What Other People Think?*.

On the twentieth anniversary of Feynman's death, composer Edward Manukyan dedicated a piece for solo clarinet to his memory.[66] It was premiered by Doug Storey, the principal clarinetist of the Amarillo Symphony.

Between 2009 and 2011, clips of an interview with Feynman were used by composer John Boswell as part of the Symphony of Science project in the second, fifth, seventh, and eleventh installments of his videos, "We Are All Connected", "The Poetry of Reality", "A Wave of Reason", and "The Quantum World".[67]

In a 1992 *New York Times* article on Feynman and his legacy, James Gleick recounts the story of how Murray Gell-Mann described what has become known as "The Feynman Algorithm" or "The Feynman Problem-Solving Algorithm" to a student: "The student asks Gell-Mann about Feynman's notes. Gell-Mann says no, Dick's methods are not the same as the methods used here. The student asks, well, what are Feynman's methods? Gell-Mann leans coyly against the blackboard and says: Dick's method is this. You write down the problem. You think very hard. (He shuts his eyes and presses his knuckles parodically to his forehead.) Then you write down the answer." [68]

In 1998, a photograph of Richard Feynman giving a lecture was part of the poster series commissioned by Apple Inc. for their "Think Different" advertising campaign.[69]

In 2011, Feynman was the subject of a biographical graphic novel entitled simply *Feynman*, written by Jim Ottaviani and illustrated by Leland Myrick.[70]

In 2013, the BBC drama *The Challenger* depicted Feynman's role on the Rogers Commission in exposing the O-ring flaw in NASA's solid-rocket boosters (SRBs), itself based in part on Feynman's book *What Do You Care What Other People Think?*[71][72]

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11.2 Textbooks and lecture notes

The Feynman Lectures on Physics is perhaps his most accessible work for anyone with an interest in physics, compiled from lectures to Caltech undergraduates in 1961–64. As news of the lectures' lucidity grew, a number of professional physicists and graduate students began to drop in to listen. Co-authors Robert B. Leighton and

Matthew Sands, colleagues of Feynman, edited and illustrated them into book form. The work has endured and is useful to this day. They were edited and supplemented in 2005 with "Feynman's Tips on Physics: A Problem-Solving Supplement to the Feynman Lectures on Physics" by Michael Gottlieb and Ralph Leighton (Robert Leighton's son), with support from Kip Thorne and other physicists.

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- *The Pleasure of Finding Things Out: The Best Short Works of Richard P. Feynman*, edited by Jeffrey Robbins, Perseus Books, 1999, ISBN 0-7382-0108-1.
- *Classic Feynman: All the Adventures of a Curious Character*, edited by Ralph Leighton, W. W. Norton & Co, 2005, ISBN 0-393-06132-9. Chronologically reordered omnibus volume of *Surely You're Joking, Mr. Feynman!* and *What Do You Care What Other People Think?*, with a bundled CD containing one of Feynman's signature lectures.
- *Quantum Man*, Atlas books, 2011, Lawrence M. Krauss, ISBN 978-0-393-06471-1.
- "Feynman: The Graphic Novel" Jim Ottaviani and Leland Myrick, ISBN 978-1-59643-259-8.

11.4 Audio and video recordings

- *Safecracker Suite* (a collection of drum pieces interspersed with Feynman telling anecdotes)
- *Los Alamos From Below* (audio, talk given by Feynman at Santa Barbara on February 6, 1975)
- *Six Easy Pieces* (original lectures upon which the book is based)
- *Six Not So Easy Pieces* (original lectures upon which the book is based)
- The Feynman Lectures on Physics: The Complete Audio Collection

- Samples of Feynman's drumming, chanting and speech are included in the songs "Tuva Groove (Bolur Daa-Bol, Bolbas Daa-Bol)" and "Kargyraa Rap (Dürgen Chugaa)" on the album *Back Tuva Future, The Adventure Continues* by Kongar-ool Ondar. The hidden track on this album also includes excerpts from lectures without musical background.
- The Messenger Lectures, given at Cornell in 1964, in which he explains basic topics in physics. Available on Project Tuva for free (See also the book *The Character of Physical Law*)
- Take the world from another point of view [videorecording] / with Richard Feynman; Films for the Hu (1972)
- The Douglas Robb Memorial Lectures Four public lectures of which the four chapters of the book QED: The Strange Theory of Light and Matter are transcripts. (1979)
- The Pleasure of Finding Things Out on YouTube (1981) (not to be confused with the later published book of same title)
- Richard Feynman: Fun to Imagine Collection, BBC Archive of 6 short films of Feynman talking in a style that is accessible to all about the physics behind common to all experiences. (1983)
- *Elementary Particles and the Laws of Physics* (1986)
- Tiny Machines: The Feynman Talk on Nanotechnology (video, 1984)
- Computers From the Inside Out (video)
- Quantum Mechanical View of Reality: Workshop at Esalen (video, 1983)
- Idiosyncratic Thinking Workshop (video, 1985)
- Bits and Pieces — From Richard's Life and Times (video, 1988)
- Strangeness Minus Three (video, BBC Horizon 1964)
- No Ordinary Genius (video, Cristopher Sykes Documentary)
- Richard Feynman — The Best Mind Since Einstein (video, Documentary)
- The Motion of Planets Around the Sun (audio, sometimes titled "Feynman's Lost Lecture")
- Nature of Matter (audio)

12 See also

- List of things named after Richard Feynman
- Feynman diagram
- Feynman checkerboard
- Flexagon
- Foresight Nanotech Institute Feynman Prize
- List of physicists
- List of theoretical physicists
- Morrie's law
- Negative probability
- One-electron universe
- Stückelberg–Feynman interpretation
- Wheeler–Feynman absorber theory

13 Notes

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15 Further reading

15.1 Articles

- *Physics Today*, American Institute of Physics magazine, February 1989 Issue. (Vol.42, No.2.) Special Feynman memorial issue containing non-technical articles on Feynman's life and work in physics.

15.2 Books

- Brown, Laurie M. and Rigden, John S. (editors) (1993) *Most of the Good Stuff: Memories of Richard Feynman* Simon and Schuster, New York, ISBN 0-88318-870-8. Commentary by Joan Feynman, John Wheeler, Hans Bethe, Julian Schwinger, Murray Gell-Mann, Daniel Hillis, David Goodstein, Freeman Dyson, and Laurie Brown

- Dyson, Freeman (1979) *Disturbing the Universe*. Harper and Row. ISBN 0-06-011108-9. Dyson's autobiography. The chapters "A Scientific Apprenticeship" and "A Ride to Albuquerque" describe his impressions of Feynman in the period 1947–48 when Dyson was a graduate student at Cornell

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- Milburn, Gerard J. (1998) *The Feynman Processor: Quantum Entanglement and the Computing Revolution* Perseus Books, ISBN 0-7382-0173-1

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- Ottaviani, Jim and Myrick, Leland (2011) *Feynman*. First Second. ISBN 978-1-59643-259-8 OCLC 664838951.

15.3 Films and plays

- *Infinity*, a movie directed by Matthew Broderick and starring Matthew Broderick as Feynman, depicting Feynman's love affair with his first wife and ending with the Trinity test. 1996.

- Parnell, Peter (2002) "QED" Applause Books, ISBN 978-1-55783-592-5, (play).

- Whittell, Crispin (2006) "Clever Dick" Oberon Books, (play)

- "The Pleasure of Finding Things Out" on YouTube. Feynman talks about his life in science and his love of exploring nature. 1981, BBC Horizon. See Christopher Sykes Productions.

- "The Quest for Tannu Tuva" on YouTube, with Richard Feynman and Ralph Leighton. 1987, BBC Horizon and PBS Nova (entitled "Last Journey of a Genius").

- "No Ordinary Genius" A two-part documentary about Feynman's life and work, with contributions from colleagues, friends and family. 1993, BBC Horizon and PBS Nova (a one-hour version, under the title "The Best Mind Since Einstein") (2 × 50 minute films)

- The Challenger (2013) A BBC Two factual drama starring William Hurt, tells the story of American Nobel prize-winning physicist Richard Feynman's determination to reveal the truth behind the 1986 space shuttle Challenger disaster.

- The Fantastic Mr Feynman. One hour documentary. 2013, BBC TV.

16 External links

- Official website